Presentation #3: MEASURING THE CIRCUMFERENCE

1. Show again the decagon and the circle. Recall the calculation of the perimeter of the decagon.

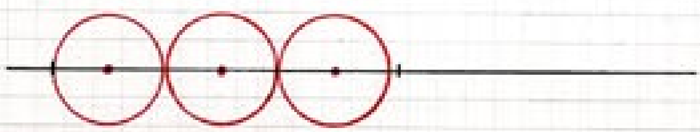
1. We know how to calculate the perimeter measurement of this polygon. The length of the side X the number of sides: s X n = P

2. Give historical references to the search for "pi." Archimedes calculated the value of "pi" between 3 1/7 and 3 10/71, a remarkable calculation for his times.

2. How do we calculate the circumference? No one, in all the historical exploration of man's mathematical work, has been able to do it exactly.

3. THE SIMPLEST METHOD: Trace a straight line on the board. Take the LARGEST CIRCLE (of 10 cm. diameter) and make a mark at a certain point on the circumference. Mark also a point on the line traced. Put the marks together. Then carefully ROLL THE CIRCLE ALONG THE LINE. And mark the point on the line where the mark on the circle again touches the line. . . This line segment is the measure of the circumference. We know that the diameter of this circle is 10 cm. NEXT WE WANT TO FIND OUT HOW MANY DIAMETERS OF THIS CIRCLE ARE CONTAINED IN THIS LINE SEGMENT.

3. We could take a string, put it around the circle and then measure the string. But this inaccurate method has another defect: it will be the measure of only one circle. Every circle will have a circumference of a different measure. We CAN DO SIMILAR MEASUREMENT THAT WILL GIVE US A COMPARISON BETWEEN THE XXXXXXXXXXXXXX CIRCUMFERENCE OF THE CIRCLE AND THE DIAMETER: What is the diameter? It is a line which crosses the circle, passing through the center.



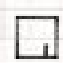
4. OBSERVATION: In the line segment we have traced three diameters and a fraction more. At this point we are not interested in that exact fraction.

5. INTRODUCE RED (additional sturdy plastic) CIRCLES, a series which includes the circles with diameters of 1, 2, 3, and 4 cm. Add those circles from the drawer which have diameters of 5, 6, 7, 8, and 9 cm. THE CHILD DOES THE ABOVE WORK WITH ALL NINE CIRCLES.

6. CONCLUSION : For each circle the diameter is contained in the corresponding circumference three times and a fraction more.

Presentation #4: Establishing the Fractional Value of that Last Part of the Line Segment

1. Returning to the circumference line of the 10 cm. circle, transfer the measure of the last fractional part to a small card.



2. Then, using that as the unit of measure, discover how many times it is contained in the diameter of one circle, as shown on the line: It is contained about 7 "and something" times in the diameter. Therefore, it has a value of about 1/7 the diameter. So we can write: 3 1/7 to express the ratio of the diameter to the circumference.

3. SECOND METHOD OF ESTABLISHING THE FRACTIONAL VALUE: A more accurate method. Divide the diameter into 100 parts. Since the diameter here is 10 cm., and the decimeter (10 cm.) corresponds to 100 mm., each part will have the value of 1 millimeter. Using our previous measuring unit on the card, we discover that it corresponds to ABOUT 14 mm. OR 14/100 of the diameter.

The constant we have now constructed is equal to 3.14. This is a better and more accurate fractional value than 3 1/7.

3.14

Presentation #4: Establishing the Fractional Value of the Line Segment. . .

4. Give the name "pi."
NOTE: There is published in the U.S. a book composed entirely of the computer-calculated digits of pi. . .an interesting item for the children, emphasizing the fact that this number never finishes.

= 3.14159. . . . .

4. For this unlimited, irrational number, the Greeks chose the name "pi."
So instead of referring to the constant as 3.14, we can say "pi" and we know that we are naming a number which is about 3.14.
We have discovered TWO DECIMAL NUMBERS of this unlimited decimal number--- there are many many more digits.

NOTE: What numbers do we use to calculate π ? We can make a big circle, then consider that polygon (of increasing numbers of sides) that is inscribed in that circle and the polygon circumscribed about it. The two provide the limits for the fractional calculation; the greater number of sides moving closer and closer to the actual value of π .

Presentation #5: Calculating the circumference

1. The problem: How can I calculate the circumference?
THE STORY OF THE PYTHAGOREAN CIRCLE.

1. I know only the diameter of the circle. It is a line segment so I can measure it. And I have the secret key "pi" to calculate the circumference.
It was for a long time a real secret: the secret of Pythagorus and his friends.

2. Lay out all of the circles in order, vertically on the mat, diameters 1 - 10.

2. We know that the diameter is contained in the circumference π times.
So we repeat the diameter 3 + (.14) times.

3. Taking the circles one at a time, ask the children to give the circumference: writing in their notebooks. . .

3. With the diameter of the circle = 1, what is the circumference?
1 X 3.14 = 3.14
With the diameter of 2: 2 X 3.14 = 6.28
With the diameter of 3. . . . . .

4. Conclusion: We have built another set of fixed numbers that represent the multiples of π . (That constant π X 2 is particularly important.) BECAUSE. . .If we know the radius of the circle, we must multiply by 2 and then times π . OR r X 2 (6.28).

5. Compose the formula for the measurement of the circumference on the mat with symbol and sign cards, INTRODUCING THE SIGN .

$$c = d \times 3.14$$
$$c = d \times \pi$$
$$c = d \cdot \pi$$
$$c = d\pi$$

6. Transform to:

$$c = 2r\pi$$
$$c = 2\pi r$$

In the formula of $2\pi r$, we recall a vast history, images, and the sound of the phrase: $2\pi r$.

6. Instead of d, we can have (2r).
We read: The circumference is equal to "2 pi r." OR With our new constant of 2π , we have 6.28 X r.

EXERCISES: Invite the children to measure the circumference of all the circles of the material, using also additional circles contained in the box of additional figures. (Red circles, triangles and squares contained in three divisions.) Invite them to do the same with circles of their own invention.

Part II: The AREA OF THE CIRCLE

Materials

1. The plane inset figures of the decagon and the circle.
2. An envelope containing:
   a) Two whole circles (green and yellow) with the circumferences marked in heavy black line and one radius drawn. (10 cm. diameter)
   b) 10 yellow tenths, corresponding to the yellow circle.
   c) 10 green tenths, all sectors but one are whole tenths, that one being divided by a radius in half.
   d) Four rectangles, the height of which is equal to the radius of the circle; the base equal to the circumference of the circle: two yellow, two green.

Presentation: Calculating the Area of the Circle

1. Show the two plane figures of the decagon and the circle. On the decagon show "A" and then form the formula on the mat with cards: $A = \frac{p \times a}{2}$

   Show, then, "A" on the circle figure. We want to show here the identity of the nomenclature between the polygon and the circle as we did with the lines.
   Write the formula for the area of the circle on the backs of the cards giving the decagon formula.

1. We know the formula for the calculation of the area of the decagon:

   $$A = \frac{p \times a}{2}$$

   In the circle the perimeter has a special name: the circumference.
   And the apothem is called the radius.
   Let's see if the formula is the same:

   $$A = \frac{c \times r}{2}$$

   NOW WE MUST PROVE IT.

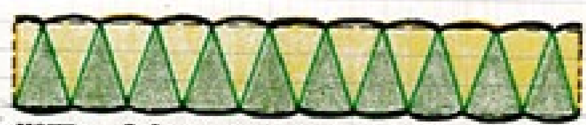
2. Take first the green circle and then the yellow, superimposing them on the red plane figure to show congruence and to establish the 10 cm. diameter.

2. We see that each of these two circles has a diameter of 10 cm. . .and is congruent with the red circle.
   The new circles have the circumference underlined and one radius is marked.

3. Introduce the sectors: construct 2 circles with them.
   SUPERIMPOSE THE WHOLE COLORED CIRCLES TO VERIFY CONGRUENCE.

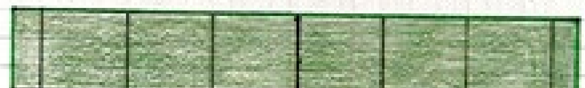
3. We see that each of these sectors is 1/10 of the circle. (And correspond to the tenths of the fraction material.)
   Our circle is divided into tenths.

4. Put the two whole circles aside, and indicate that, as in the prior area work, we begin by doubling the figure, using these two circles formed of tenths. ARRANGE THE GREEN SECTORS AS SHOWN:



NOTE: Colors reversed: 1/20 are green

5. TAKE THE YELLOW SECTORS AND PLACE THEM SO THAT the teeth of the saw disappear. (The last yellow sector is divided to complete the figure into twentieths)



The new figure (top) is equal to the circle. . .times 2.

We have more or less a rectangle because the base is formed of arcs. . .BUT I KNOW THAT THE BASE IS EQUAL TO THE CIRCUMFERENCE OF THE CIRCLE. If the sectors become smaller, the arcs become smaller . . .then the base will be formed of shorter arcs, closer to a straight line.
We KNOW THAT THIS RECTANGLE IS EQUAL TO 2 CIRCLES.

6. Superimpose the green rectangle
   Remove the rectangle of sectors and display only the whole rectangle.
   State the equality of the rectangle base and the circumference.
   Then, using a radius (limiting radius of one sector), identify the height of rectangle with the radius of the circle.

6. It is not a very accurate rectangle. But with this whole rectangle, we know that the base corresponds to the circumference of the circle. (Both the base of the rectangle and the circumference of the circles are in black.)
   And we can show that the height of the rectangle is equal to the radius of the circle.

7. Conclusions: We can say that the area of this rectangle is equal to the area of <u>two circles</u> having a radius equal to the height of the rectangle and a circumference equal to the base.

$$h_R = r_C \quad \text{AND} \quad b_R = c_C$$

So we must divide the product of (c X r) by 2:

$$A_c = \frac{c \times r}{2}$$

$$A_c = \frac{cr}{2}$$

8. Substitute the formula for "c" in the new formula for the area:

$$c = 2\pi r$$

$$A = \frac{2\pi r \times r}{2}$$

$$A = \pi r^2$$

9. NOW WE WANT TO SHOW THIS AND VERIFY IT WITH THE MATERIALS:

a) Show both the yellow and green whole circles adjacent to the long green rectangle.

a) We know that our rectangle is equivalent to the two circles.

b) FOLD THE GREEN RECTANGLE in half and remove one circle. . .the green.

b) As with our work with the yellow material, we now want to calculate the area of only ONE CIRCLE.

c) Unfold the green rectangle and introduce the two yellow rectangles, each one-half of the green; superimposing the two to verify congruence.

c) We see that each yellow rectangle is equal to one-half the green rectangle. So the two yellow rectangles are equal among themselves.

d) Remove one yellow rectangle and SHOW THE YELLOW CIRCLE ALONGSIDE THE ONE YELLOW RECTANGLE. ADD one yellow tenth sector.

d) This yellow sector will be our measurer: we can say that the sides (limiting rays) of the sector are equal to the radius of the circle.

e) Observe the black lines on the yellow rectangle.

e) The rectangle is divided into three squares + a fraction of a square.

What do the squares represent in relation to the radius of the circle?

$h_R = r_o$ (or s of the sector)

$h_R = s_S$ (side of the square)

$s_S = r_o$ (and the other side)

$A_S = r^2$

II. CONCLUSION: This rectangle is constructed of the square of the radius 3 and 0.14 times. . .OR this rectangle is formed of r^2 taken π times.

How many squares are there? 3 + (3 + 1/7 or 3.14)

III. 10. Organize the formula again:

$$A = \pi \times r^2$$

$$A = \pi r^2$$

NOTE: We have constructed the formula for the area of the circle in two ways: in the first method the important element was c (circumference); and in the second the important element was the radius (r). Which is the most useful rule?

AREA OF THE PARTS OF THE CIRCLE

Presentation #1: The Sector of the Circle

Material

1. The metal insets: circle fractions.
2. Frame #14 of the insets of equivalence.
3. The Montessori protractor.
4. Identifying paper slips (for formulas.)

Area of the Parts of a Circle. . .
Sector. . .

1. Display the series of metal insets of the circles divided into fractions: then take one aside. (Here the thirds used.)

2. Showing the circle divided into thirds, remove one third, setting that aside, and IDENTIFY THE SECTOR OF THE CIRCLE IN TERMS OF THE EMPTY AND OCCUPIED SPACES IN THE FRAME NOW. Define sector, reviewing the concept.

2. This is a sector of the circle. These remaining 2/3 also compose a sector of the circle. The sector of the circle is that figure formed by two radii and the intercepted arc of the circle.

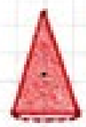
3. Repeat the experience with all the insets, removing one fraction each time. STOP WITH TENTHS AND EMPHASIZE:

3. This is a sector of the circle; and the part remaining is a sector. This sector corresp onds to 1/10 of the circle.

4. From frame #14, the decagon, take one of the triangles. Display the two pieces side by side.

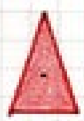
4. We have said that the circle is a regular polygon with an infinite number of sides; and the decagon is a regular polygon with ten sides.

5. SUPERIMPOSE THE TWO.



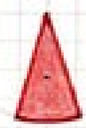
5. The two figures correspond---ALMOST exactly. One is limited by an arc, one by a straight line segment.

6. Establish the nomenclature of the two figures simultaneously:



b = base

h = height (line joining the midpoint of the side to the apex, crossing through the knob)



base = arc = a

height = that line which also crosses through the knob from midpoint to apex AND = r or the sides of the sector

7. Show the formula for the triangle on the mat with slips: then transform the formula with the new elements of nomenclature into that for the sector, writing the new terms on the back of the slips.

$$A_{\Delta} = \frac{bh}{2} \qquad A_{\Delta} = \frac{ar}{2}$$

The area of the sector of the circle is obtained by multiplying a X $\frac{r}{2}$.

8. THEN HOW DO WE CALCULATE THE ARC? I must determine what fractional unit of the circle the arc belongs to.

FIRST LEVEL: Early Elementary (?) Show the sector in the Montessori protractor to measure the fractional unit of the arc.

SECOND LEVEL: I know the circumference (c) of the circle. $c = 2\pi r$
That is, the circumference is obtained by dividing the diameter into two equal parts, giving the measure of the radius (r).
Then, if $r = 5$
$c = 10 \cdot 3.14$
We know that to multiply by 10 gives us a movement of the decimal point to the right one place. . .
so $c = 31.4$
And to calculate the arc of one-tenth of the circle, that are described by the fractional inset 1/10, we have
$1/10 \cdot c = 3.14$ (Dividing by 0.10)

So I now know that $a = 3.14$ and $r = 5$ $\quad A_{\Delta_{10}} = \frac{3.14 \times 5}{2}$

Then to calculate the sector of $\frac{2}{3}$: $A_{\Delta_{\frac{2}{3}}} = [(31.4 \cdot \frac{2}{3}) \times r] \div 2 \quad (r = 5)$

Presentation #2: The Area of the Segment of the Circle

Material

1. Circle insets divided into halves, and thirds.
2. The inset of the inscribed triangle.
3. Slips of paper for the organization of the formula.

1. From the inset of the inscribed triangle, remove one segment and identify. DEFINE.

   1. This is a segment of a circle. And this (the space still occupied in the frame) is a segment of a circle. The figure formed by a cord and its arc is called the segment of the circle.

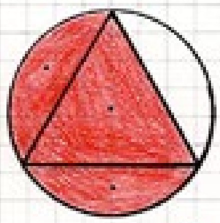
2. STATING THE PROBLEM: Note that the formula for the two segments will have to be slightly different.

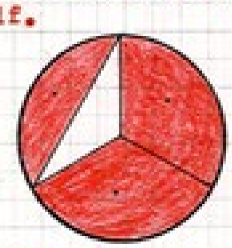
   2. We want to calculate the area of both the large and the small segments which we have identified in the frame. One is larger than 1/2 the circle and one is smaller than 1/2 the circle.

3. Examine the segment smaller than 1/2:

   a) Remove from the inset of thirds ONE THIRD AND FIT IN THE SEGMENT.

   a) We see that the area of the segment will be equal to this 1/3 of the circle MINUS this isosceles triangle which is not covered by the segment itself.





$$A_{segment} = A_{sector\ (1/3)} - A_{triangle}$$

   b) Identify the nomenclature of the triangle in terms of the circle's parts.

   b) b_t = cord that determines segment
   top vertex of the triangle is at the center of the circle
   h_t = line from the center of the circle to the midpoint of the cord

   c) Organize the formula:

$$A_{SEG_o} = A_{SECTOR_o} - A_\Delta$$

$$A_{SECTOR_o} = \frac{a \times r}{2} \qquad (a\ (\text{arc}) = l)(k = \text{cord})$$

$$A_{SEG_o} = \frac{l \times r}{2} - \frac{k \times h}{2}$$

$$A_{SEG_o} = \frac{lr}{2} - \frac{kh}{2}$$

$$A_{SEG_o} = \frac{lr - kh}{2}$$

$$A_{SEG_<} = \frac{lr \ominus kh}{2}$$

(SEG< and ⊖ indicate segment less than 1/2 o)

4. The segment greater than 1/2: REMOVE THE SEGMENT FROM THE THIRDS FRAME: compare it now with the segment as shown in the first frame (above.) We can see that the segment greater than 1/2 is equal to the two sectors (two thirds) + that same triangle:

$$A_{SEG_>} = \frac{lr}{2} + \frac{kh}{2} = \frac{lr \oplus kh}{2}$$

5. PUT BOTH FORMULAS TOGETHER:

$$A_{SEG_{o\,<>}} = \frac{lr \mp kh}{2}$$

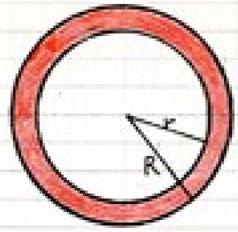
When I want the area of a segment less than 1/2 I use −.
When I want the area of a segment greater than 1/2, I use +.

Presentation #3: The Annulus

$$A_a = A_O - A_o$$

$$A_a = \pi R^2 - \pi r^2$$

$$A_a = \pi (R^2 - r^2)$$



AREA OF THE ELLIPSE

Material

1. From the plane figures of the cabinet: the ellipse and the largest circle
2. A cone and a cylinder. (Cardboard toilet tissue roll and ice cream cone)
3. Knife
4. A CLASSIFIED NOMENCLATURE (prepared as described.)

Presentation #1: What is an ellipse?

1. Begin with the cardboard cylinder: DEFINE. Give nomenclature.

   1. This is a cylinder. This is the base of the cylinder; this is the other base of the cylinder.

2. Set the cylinder on the mat and with a sheet of paper, define those planes parallel to the base and those not parallel to the base.

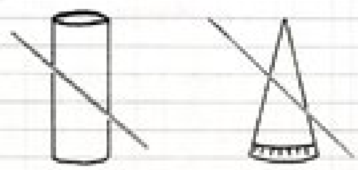
   2. This plane is parallel to this base and this base. This plane is not parallel to the bases.

3. CUT THE CYLINDER on a plane NOT parallel to the bases. Then show the paper flush between the parts.

   3. This cylinder has been cut into two parts following a plane not parallel to the bases.

4. REPEAT THE EXPERIENCE WITH THE CONE.

   This cone has been cut into two parts following a plane not parallel to its base.



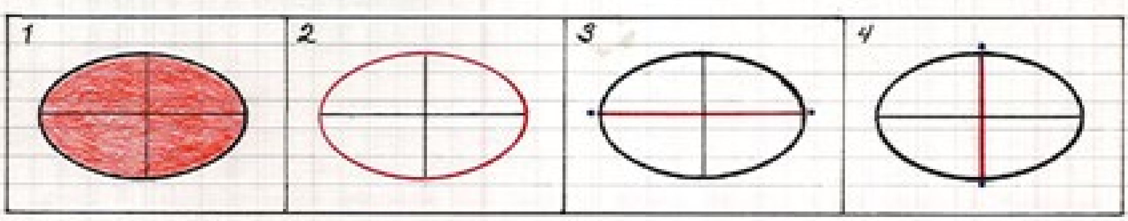
5. Display the four parts on the mat, with the cut edges aligned towards the children. INVITE THEM TO TRACE THE CUT EDGES WITH THEIR FINGERS.

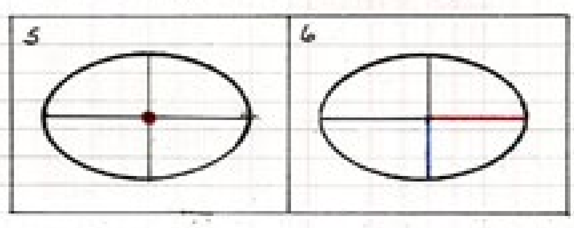
   5. This is an ellipse. This is an ellipse equal to the first. This is another ellipse. And this is an ellipse equal to it.

NOTE: An interesting study with the cylinders. Cut a series of the cylinders at graduated degrees. Each time we have a slightly different ellipse. And, when we cut an angle of 90°, we have cut a circle. The limit.

6. Using the inset of the ellipse in the frame, give the nomenclature.

7. Then present the corresponding classified nomenclature (picture cards, slips of idenfification and definitions):





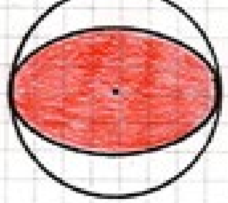
1) Ellipse
2) The conical line that limits the surface of the ellipse; the ellipse.
3) Major axis of symmetry. (Vertices)
4) Minor axis of symmetry. (Vertices)
5) Center of symmetry.
6) ½ the major axis; ½ the minor axis of symmetry. *(semi-major; semi-minor)*

Presentation #2: Calculating the Area of the Ellipse

Material

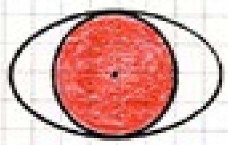
1. From the plane insets: the ellipse (figure and frame); the largest circle and the circle of 6 cm. diameter (figures and frames.)
2. A board, thumb tacks and string prepared for the ellipsograph.

1. Fit the ellipse into the frame of the largest circle: Define in terms of the circle and identify the major axis of symmetry.



1. This is an ellipse.
"Ellipse" means "Something is missing."
This implies that the ellipse is compared to another figure: the circle.
Which are the parts missing from the circle? These uncovered parts.
So from the ellipse something is missing from the perfect figure which is the circle.
The major axis of symmetry is equal to the diameter of the circle.

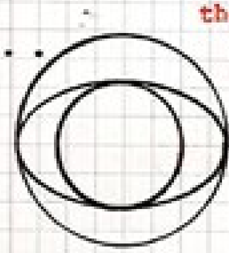
2. Take the circle of 6 cm. diameter and inscribe it in the frame of the ellipse.



NOTE: The concept of the prolate circle is an important one for later volume work; the concept of the prolate spheroid.

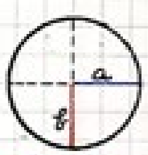
2. The ellipse is still an imperfect figure.
Now it is larger than the circle.
The fact that it is smaller than the circle gives it the name ellipse.
The fact that it is larger than the circle gives it another special name: the PROLATE CIRCLE.
(From pro + late (fero): Latin meaning forwards + to bring or to push.)
So the circle whose side is pushed forwards gives us an elongated-sided figure, a stretched circle.
The minor axis of symmetry is equal to the diameter of this circle.

3. In a drawing, combine the two. . .

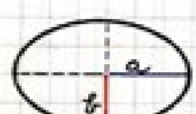


. . .We see that, regarding the ellipse, there are two circles involved.

4.



Compare the nomenclature of the circle and the ellipse:



Center

The circumference is the closed curved line line equidistant at every point from the center.

Center of symmetry

The conic plane, that closed curved line which limits the ellipse, is also called the circumference.

The diameter of the circle can be compared to the major or minor axes of symmetry.

Then. . .the radius	=	semi-major axis of symmetry.
and another radius	=	semi-minor axis of symmetry.

HERE IS THE DIFFERENCE BETWEEN THETWO FIGURES: The radii of the circle are equal; the semi-major axis of symmetry and the semi-minor axis of symmetry are not.

5. Organizing the rule: If we identify the major axis of symmetry as 2a, then we have a semi-major axis of a. Identifying the minor axis as 2b, we have a semi-minor axis of b.
THE RADII ARE SEMI-AXES OF SYMMETRY OF THE CIRCLE. We can identify them also as a and b.
BUT: in the circle $a = r$
$b = r$
so ...$a = b$

6. Organize the formula: We must begin our calculation of the area of the ellipse with what we know. . .the area of the circle

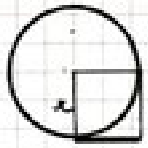
$$A_o = \pi r^2$$
$$A_o = \pi \times r \times r$$

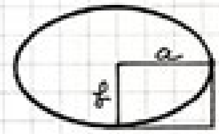
$$A = \pi \times a \times b$$
$$A_e = \pi\ ab$$

We have organized the formula for the area of the ellipse: we multiply π X the lengths of the semi-axes.

7. We can do the OPPOSITE WORK AS A PROOF: To calculate the area of the ellipse we have used two circles. The semi-axes are the radii of the two circles; that is, they are the radii of two different circles.

Now we can find the area of the ellipse and then calculate the area of the circles: we think of the circle, then, as a particular case of the ellipse. It is, in fact, the limit, the perfection of the ellipse. As the square is to the rectangle, so is the circle to the ellipse. That is, the circle is the perfection of the set of ellipses.





Beginning with the formula for the area of the ellipse, the area of the closed figure which is the limit of the ellipses is:

$$A_o = \pi\ ab$$

In that figure, the semi-axes are equal, so

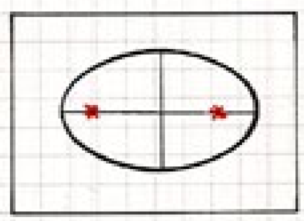
$$a = b$$

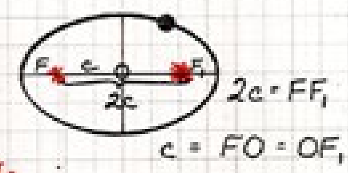
Because a = b, we can designate r X r, and

$$A_o = \pi r^2$$

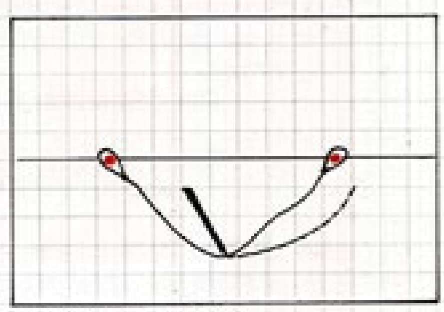
Constructing the Ellipse

1. Add a card to the nomenclature, identifying two points along the major axis of symmetry. We will call them STARS. . .or FOCI (focus). We designate the distance between these notable points 2c. Then c = $\frac{1}{2}$ the distance between the points which we call F and F_1. NOTE: It is Kepler who identifies these points for us; and from them comes the FIRST LAW: the orbit followed by the earth (or planet) is an ellipse where the sun occupies one of the two foci. (We can imagine the earth traveling the orbit of our ellipse with the sun as one of those foci.)





2. INTRODUCE THE SIMPLE ELLIPSOGRAPH: The method of the string, or the method of the gardener.
   a) Trace a line horizontally on the board.
   b) Identify two point along the line with thumb tacks.
   c) Prepare a string, the total length of which, from the end of one loop to the end of the other (as shown) is greater than 2c; that is, the total length of the string will equal the major axis of symmetry.
   d) Knot the string, as shown to form the two loops. The knots will be equal to the two foci on the string.
   e) With a pencil, trace the line given by the lack in the string.



3. With this ellipsograph string we have traced a special ellipse. Its axes are 40 and 24 respectively; that is, four times that of the ellipse of the plane figures with axes of 10 and 6. So we have similar ellipses.

$A_e = \pi\ ab$ | $A_E = \pi\ ab$
$= \pi\ 5 \cdot 3$ | $= \pi\ 20 \cdot 12$
$= \pi\ 15$ | $= \pi\ 240$

The ratio of the lines is 1:4.
The ratio of the area is 1:16.

A SUMMARY OF THE WORK OF AREA: The Tiling Game

Material

1. A board 60 X 40 cm^2 which is the pavement; bottom covered in grey heavy paper.
2. A box containing 6 examples of each of the following figures present in the cabinet (in heavy bright colored paper):

equilateral triangles - red	heptagons - pink
squares - yellow	octagons - moss green
rectangles - dark brown	nonagons - bright orange
rhombii - light green	decagons - tan
pentagons - bright green	first flower - grey
hexagons - turquoise	second flowers - red-orange

...some other figures....

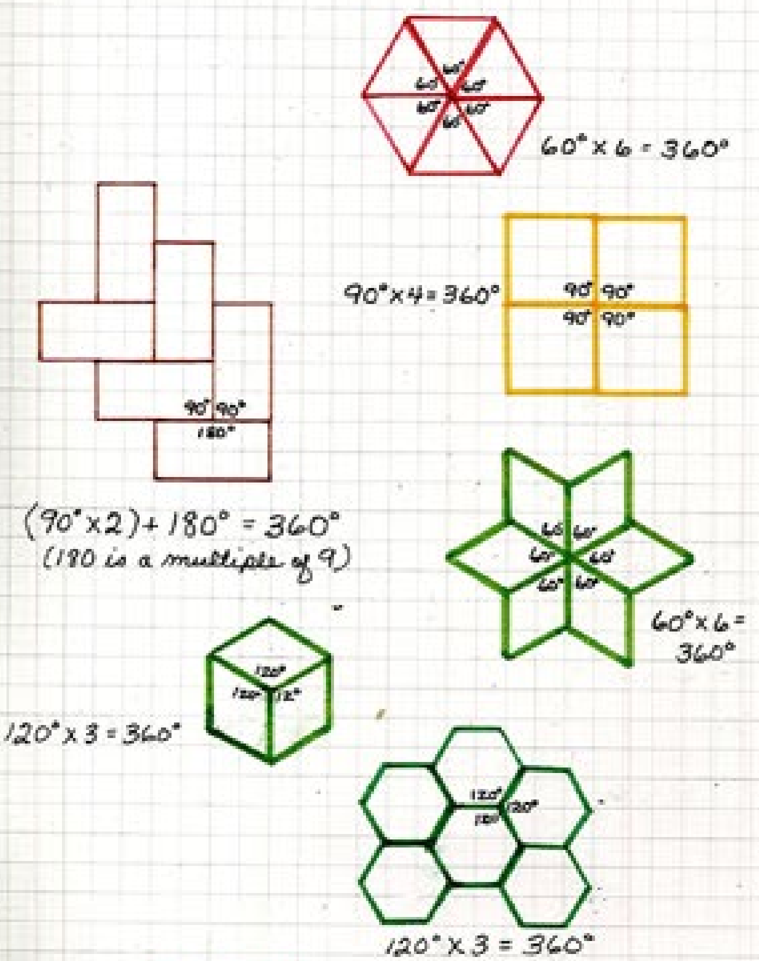
Presentation: A Pythagorean problem

1. Present the problem: We must pave a room. We want to discover with which figures here it is possible to cover a surface completely.

2. Begin with the triangles, taking all the available triangles and arranging in a pattern to discover whether or not the triangles will cover a surface. We discover that it is possible to cover a surface with equilateral triangles.

3. Proceed through the other figures, placing them on the pavement in various patterns to discover whether or not it is possible to pave a surface with them: the result is that we discover only five figures with which it is possible: the equilateral triangle, the square, the rectangle, the rhombii, and the hexagon.

4. Line this five figures up horizontally on the pavement and analyze the reasons:



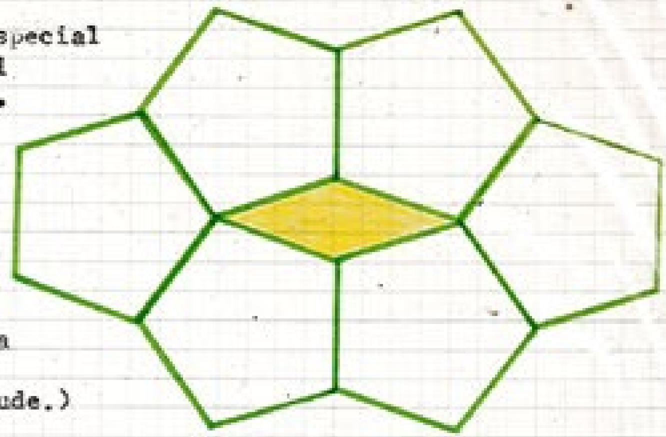
4. Why can we cover a surface with only these figures?
We have studied the interior angles of the polygon:
We know that the interior angles of the triangle are equal to 180°... and that in the equilateral triangle each angle is equal to 60°
If we use the equilateral triangle, we need six triangles (60° X 6) to measure a whole angle of 360°
The triangle has angles which are contained exactly 6 times in the whole angle.
The angle of the square is 90°---
Then with four squares we have a whole angle: the angle of the square is contained 4 times in a whole angle.
The same is true for the rectangle.
The rhombus has two angles of 120° and two angles of 60°---the angles are contained perfectly in the whole angle; and we can combine them in several different ways to form this whole angle.
What are the angles (interior) of the hexagon?
The sum of the interior angles of the polygon: $180° \times (n - 2)$
$= 180° \times 4 = 720°$
$720° \div 6 = 120°$
The bees use this wonderful combination of 120° X 3 to form the whole angles that cover the surface of their hives.

5. CONCLUSION: I can pave a surface with the equilateral triangle, the square, the rectangle, the rhombus, and the hexagon because their angles are contained an exact number of times in the whole angle.

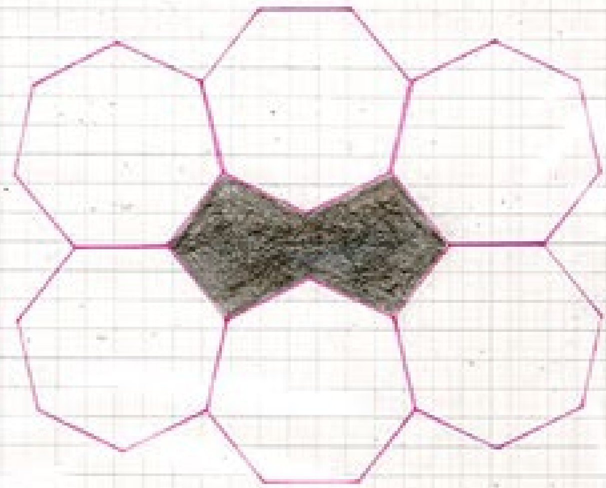
6. Another Problem: If we want to cover a surface with the rest of the polygons, we must use special other figures:

A. Regular Polygon: We need a special rhombus. The sides are equal to the sides of the pentagon. We can see that:
Obtuse angles = $360^{o} - 2(108^{o})$
Acute angles = $360^{o} - 3(108^{o})$

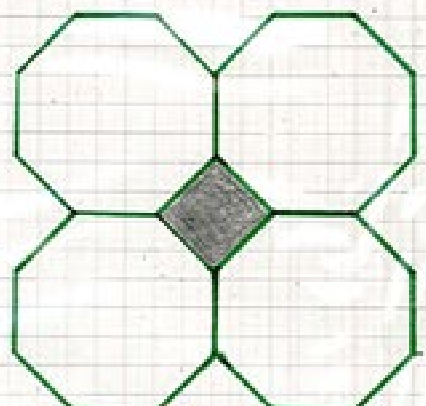
(Each angle of the regular pentagon = 108^{o}; our formula for the sum of the interior angles gives us this amplitude.)



B. Regular Heptagon: The butterfly; formed by joining two pentagons along one side. Because 360^{o} is not divisible by seven, the calculation of the degrees which give us this figure is more difficult.

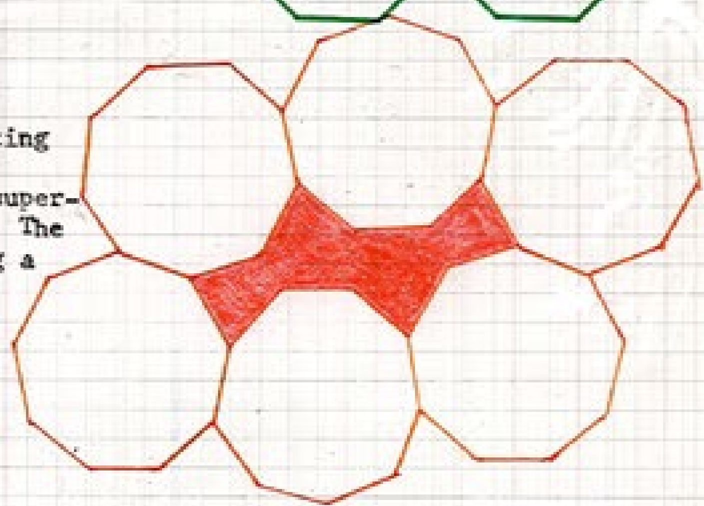


C. Regular Octagon: The necessary extra figure is the square with the side equal to the side of the octagon. A pattern which has a vast application in the field of architecture.



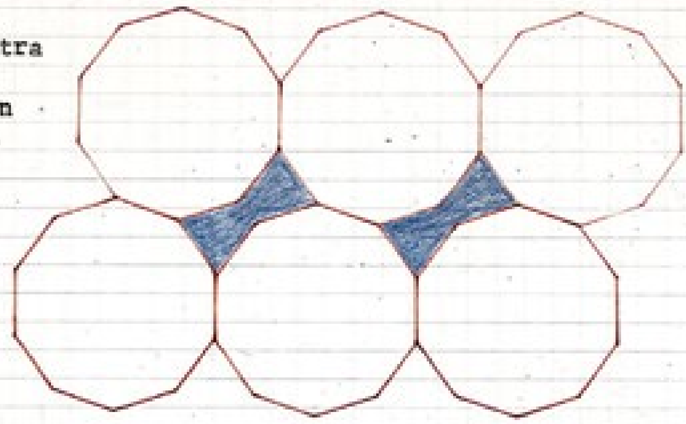
D. Regular Nonagon: We need an irregular dodecagon, each side of which is equal to the. . .

. . . . .side of the nonagon. This interesting figure can be cut in half, one half turned over, and superimposed on the other half. The two halves are joined along a side that is not equal to the side of the nonagon.

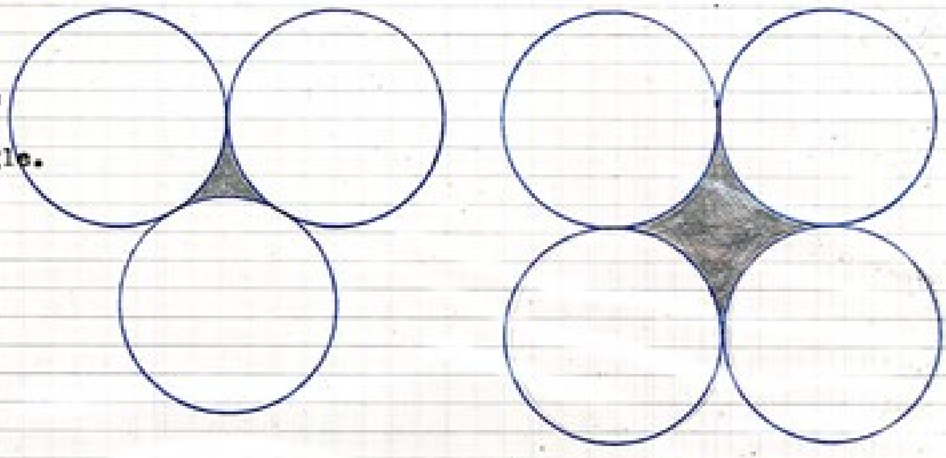


6. . . Other figures. . .

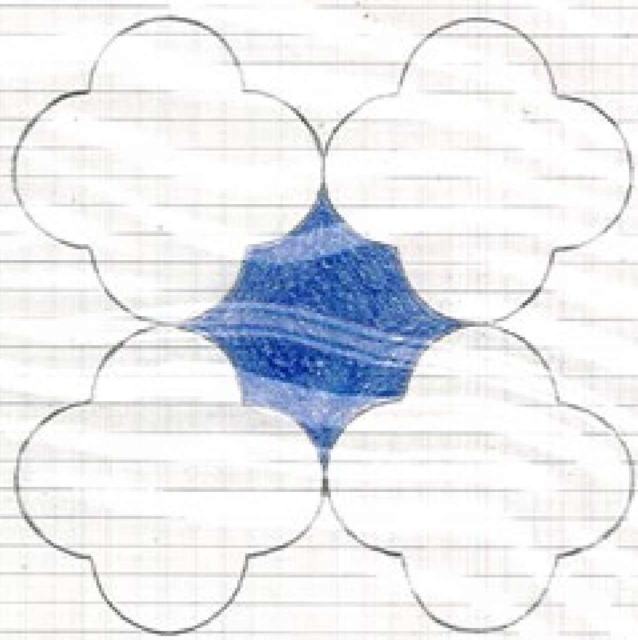
E. **Regular Decagon:** The extra figure necessary is an irregular concave hexagon with sides equal to the side of the decagon. It is formed of two equal trapezoids united along one side that is not equal to the side of the decagon.



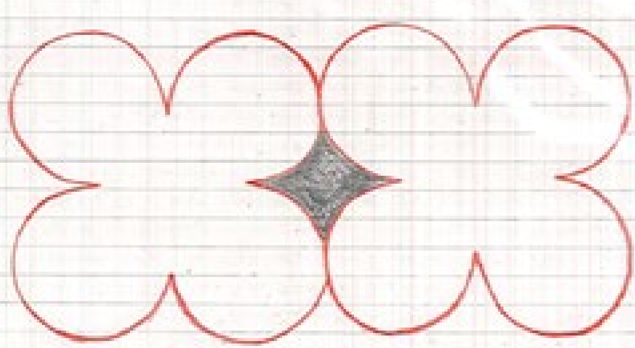
F. **Circle:** Two possibilities:
Curvalinear square
or
Curvalinear triangle.



G. **First flower:** A curvalinear octagon.



H. **Second flower:** A curvalinear square; because the second flower is formed of four small circles.



THE FOURTH PLANE: THE STUDY OF THE AREAS

INTRODUCTION: The Difference Between the Concept of Surface and the Concept of Area

We know that the area is not equal to the surface. In fact, the surface is that part of the plane limited by a closed line: a curved or a broken closed line. As a result we have two big groups: polygons and the circle group: ellipse, oval, etc. In this sense all of the plane insets represent surfaces. It is precisely that thin layer of paint which covers each.

The area is a measure. Without any relation to any specific system. What is the relationship between the area and the surface? The area is the measuring of the surface. Thus it cannot be said "calculate the surface." But rather "calculate the area of the surface." Simplifying it we may say "calculate the area of the (square.)" (As opposed to "calculate the area of the surface of the (square).")

Is the child prepared to face these concepts? Yes. He has had an indirect as well as a direct preparation for both concepts. In regard to surface: 1) the presentation of the red cardboard square presented in conjunction with the four fundamental concepts in geometry and 2) the hundred-square. Thus he has met surface in both his geometry work and in his many experiences with the decimal system---directly.

He has been prepared for area through multiplication. When he represents the multiplication 9 X 3, the child takes the bar of 9 three times. And when he was asked to say how many beads there were, he showed that result with 27 beads; that is, two ten-bars and one seven bar. (generally shown in vertical columns as the product as opposed to the horizontal surface he creates with the multiplication representation.) The surface was any group of bars showing a multiplication, and the area was the beads forming the result.

Material: "Material of Area" OR "The Yellow Material

A box containing 20 pieces. Included are:

a) Four rectangles
b) Two common parallelograms, the second composed of two pieces.
c) Fourteen pieces which, when combined, give a certain number of triangles which we classify by their angles only.
   1) 3 Acute-angled triangles.
   2) 2 Right-angled triangles.
   3) 2 Obtuse-angled triangles.

Presentation #1: The Four Rectangles: Level One: How to Measure a Surface

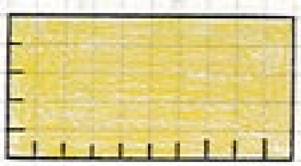
1. Showing only the reverse side of the rectangle marked with only very short lines along two sides, give the NOMENCLATURE OF THE RECTANGLE. (Refer to Classified Nomenclature definitions.) NOTE: Why do we begin with the rectangle, having shown the rectangle as the second term in every case of the insets of equivalence?

1. What is this? A rectangle. The part of the plane enclosed inside the rectangle is called the surface. Sides? Base? Perimeter? Angles? Vertices? Altitude? Diagonal?

2. Propose the question of measuring the surface. Take a piece of paper (shown as a rectangle) and MARK A SEGMENT OF MEASURE. It must be 2 centimeters although we say that it is a random measure. CUT on that mark, giving a tool of measure.

2. How will I measure the surface of this rectangle? I must begin by deciding on a unit of measure. What SHAPE will our measure have that measures the surface? Let's agree on this length of measure. It is not a centimeter. It is not an inch. It is OUR MEASURE.

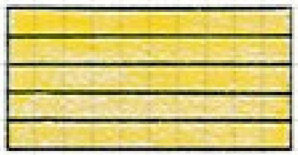
3. Begin with the long side, marking off the units of the chosen measure. Then mark a short side. REVERSE THE RECT. :



3. Let's transfer our measure to two consecutive sides. . .of our rectangle. NOW OUR RESULT IS THIS.

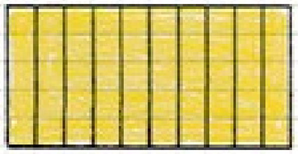
We have ten divisions on one side made with our nine marks. . .and five divisions on the short side made with four marks. But WE STILL DO NOT HAVE ENOUGH EVIDENCE TO DETERMINE OUR UNIT FOR MEASURING THE SURFACE.

4\. Prolonging the marks on the long side, we show rectangle #2:



4\. When we prolong the marks on the long side, our result is this rectangle. It seems that my unit of measure is one of these ten rectangles. . .NO

5\. Show the first board again with only the marks and the progression to the third rectangle:

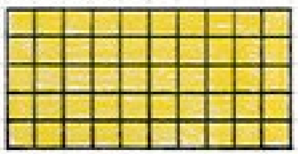


5\. We have to prolong the lines of the short side, too. Now we obtain five long rectangles. Is this my unit of measure? NO.

6\. Show the first three rectangles together:

#1 #2 #3

Introduce rectangle #4 as a result of superimposing #2 and #3.



6\. Let's create the unit of measure. We superimpose the long rectangles (few) on the short rectangles (many). I cannot see through the top rectangle here, but let's imagine that it is transparent. The result obtained is a series of SQUARES. Each SQUARE is the unit of measure of our surface. I CAN PUT THE FIRST THREE RECTANGLES ASIDE--- I work with the last one!!

7\. Emphasize that the square may be ANY SQUARE.

7\. It is important to know that the unit of measure is A square: not this square, but ANY square. Any square can measure any surface.

8\. COUNT THE SQUARES: It is important to actually count the squares while the child watches.

8\. What is the area of this surface? Is there a way we can express the measurement of this surface? Let's count the squares. 50

9\. USE BEAD BARS TO REPRESENT THE SAME CALCULATED SURFACE:

A. First show five bars of 10. Show the bars as horizontal rows and count the bars by tens.

B. Then take ten bars of 5, counting this time the VERTICAL COLUMNS by fives.

9\. It is boring to count squares to calculate surfaces. Is there another way? Let's take five bars of 10. IT IS THE SAME: A SURFACE. But here the surface is composed of beads instead of squares. Both show 50 units!!! To know how many beads, did we count them? NO---how did we know there were 50? I counted the long rows: 10, 20, 30. . . I COULD DO THE SAME THING WITH THE BARS OF 5. Now we will count the short columns by fives: 5, 10, 15, 20 . . .

10\. INTRODUCE THE FACTORS OF 50. We see then, with the child, that two factors of 50 are found on the rectangle: the number of squares on the two consecutive sides.

There MUST be an easier way. What are the factors of 50: 50 is formed by 5 X 10. (That's here in our surface of bead bars.) AND 50 is formed of 10 X 5. (That's here in the bead surface, too)

SO. . .we have said that 50 is formed of 10 X 5 OR 5 X 10. Let's count the divisions of the long side: 10. The short side: 5. It's here: the factors of 50. 10 X 5: that's 50.

11\. CONCLUSION: I have found a way to calculate the area of the surface. I have to multiply the divisions of one side times the divisions of the other side.

12\. GIVE THE PROPER NOMENCLATURE TO BASE AND ALTITUDE OF THE RECTANGLE AND RESTATE THE RULE:

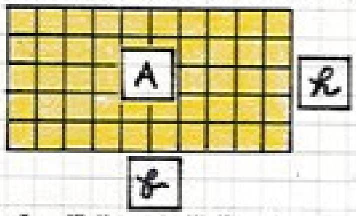
12\. The area of the rectangle is obtained by multiplying the measures of the base times the measures of the altitude.

Presentation #2: Second Level: Constructing the Formula

1. Show the corresponding symbols for terms used in the preceeding presentation, showing a series of slips on the mat:
   In place of area, we can write. . . . . A
   In place of base. . . b
   h

   We also need: the operations symbols: = X

2. Show all the symbols for the terms on the rectangle:



All of this is the surface. We show the symbol A for area.

3. Reverse the symbols "b" and "h", showing the base as both long sides and the altitude as both short sides.

   3. This can also be named the base. . . and this the altitude.

4. WRITE THE FORMULA WITH THE SYMBOLS. Show the standard writing of the formula.
   NOTE: All of this is done on the mat with the paper slips.

   4. I know that A = 50, b = 10 and h = 5.
   I know that 50 = 10 X 5.
   SO. . . A = b X h
   That can be simplified to A = b · h
   and further to A = bh.

Presentation #3: Third Level: Inverse Properties: Corollaries

1. FIRST INVERSE CASE: Show, in place of the symbol "h" as shown in the figure above, a slip "? "

   1. If I know the value of A and the value of b, what will "h" be?

2. Ask the child to list the special cases of multiplication, using two random numbers first and then transfering all the numbers to the corresponding calculation: 50 = 10 X 5.

   2. 10 X ? = 50
   ? X 5 = 50
   ? = 10 X 5
   50 = 10 X ?
   50 = ? X 5
   ? X ? = 50

   AND 10 X 5 = 50
   We have already used this case---so we can leave it out.

3. Specify that case which will fit the missing element in this problem. READ the statement, including the missing element so that the CHILD CAN FIND THE CORRESPONDING SPECIAL CASE.

   3. I have 50 squares; I know they are the result of 10 (the base) times ?
   I don't know what that height is.
   SO: Fifty is the product of 10 taken how many times?

   Special case: 50 = 10 X ?

4. Show the calculation which represents the MENTAL OPERATION DONE TO SOLVE THE SPECIAL CASE.

   4. How did we solve this missing element?
   50 = 10 X ?   50 ÷ 10 = 5

5. THEN SHOW ON THE MAT THE SAME OPERATION WITH THE SYMBOLS.

   h = A ÷ b

   $h = \frac{A}{b}$

6. SECOND INVERSE CASE: Now we show the question mark where the symbol "b" has been. We identify the special case: A = ? X h OR 50 = ? X 5. We can read that: Fifty is obtained by taking what number 5 times? Then our mental calculation is: 50 ÷ 5 = 10 OR A ÷ h = b OR $b = \frac{A}{h}$

   SHOW THIS WITH THE SYMBOLS ON THE MAT: To form the second inverse rule, it is enough to switch the b and h.

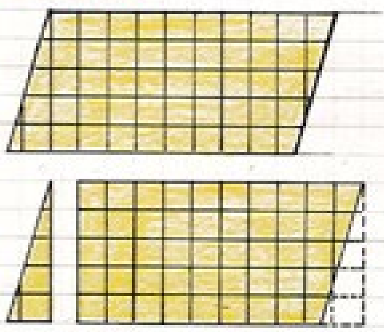
7. THIRD INVERSE CASE: Here we have ONLY THE VALUE OF A: we show question marks in place of both b and h. Now we have the special case 50 = ? X ? We can show that 50 is obtained in many different ways, but only 10 X 5 and 5 X 10 will work here. It is evident with the material.

Presentation: The Area of the Parallelogram (Common)

I. 1. IDENTIFICATION OF THE FIGURE.

1. A common parallelogram

II. 2. COUNTING THE SQUARES: Superimpose a second "identical" parallelogram on the first. Then remove the first one---we use the second as the mediator.



2. We discover that we cannot count all the squares because some are incomplete. How can I find out how many there are? By magic---of course. We superimpose an identical parallelogram on the first. In reality, though, this is not the same. It is formed of two parts---divided by a perpendicular traced to the longer side. We can separate the two parts. These two parts constitute a mediator that allow us to pass from the common parallelogram to another figure---A RECTANGLE. Now can we count the squares? YES

3. Superimpose the actual rectangle on the two-part rectangle, as formed above: VERIFY CONGRUENCE.

4. CONCLUSION OF EQUIVALENCE: the sensorial experience of equivalence.

4. Now I can say that this common parallelogram is equivalent to the rectangle because I have transformed the parallelogram to a rectangle.

III. 5. ORGANIZATION OF THE RULE FOR THE CALCULATION OF THE AREA OF THE PARALLELOGRAM. Using the rectangle and the common parallelogram figures, show by juxtaposition that both bases and heights are the same.

5. How do we calculate the area of the common parallelogram? The same as we do for the rectangle. Why? Because they have the same base and height.

CONCLUSION: To calculate the area of the common parallelogram, we multiply the base times the height.

IV. 6. ORGANIZE THE FORMULA: show with symbols on the mat: $A = bh$

V. 7. STATE AND ORGANIZE THE INVERSE FORMULAS: With the children, do several examples of the inverse cases, showing the missing element with a question mark on the actual figure for clarification.

If I know the area (A) and the base (b), how do I calculate the height (h) ?

$? = A/b \qquad h = A/b$

How do I calculate the base when I know the area and the height?

Presentation: The Area of the Triangle

1. Display the three acute-angled triangles which can be constructed with the pieces from the box:

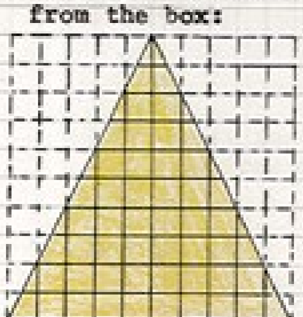


fig. 1

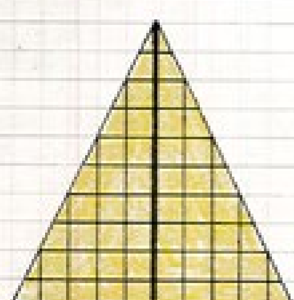


fig. 2

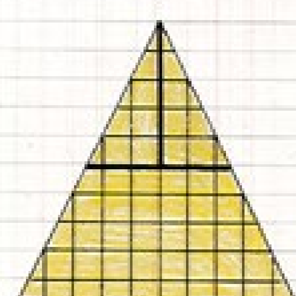


fig. 3

We note now that each is an ISOSCELES ACUTE-ANGLED TRIANGLE: *Identification.*

2. To COUNT THE SQUARES we see that a mediator is necessary:
   a) Verify congruency between the triangle in fig. 1 and those two triangles which form the triangle in fig. 2

2. . .counting the squares. . .
   b) Using those two triangles of fig. 2 (each half of the isosceles acute-angled triangle), form a square with the whole triangle as shown in fig. 1.
   c) NOW WE CAN COUNT THE SQUARES: there are so many. . .how many? 100.
   We have simply multiplied the base of the square times the height.

3. SLIDE THE WHOLE TRIANGLE DOWN OUT OF THE CENTER OF THE SQUARE. Show the formation of the square and its relation to the triangle.

3. We see that we have a square which is formed of two triangles that are equal. What is this triangle? It is 1/2 the square.

4. ORGANIZE THE RULE: I can say that the original triangle is equal to ½ the rectangle having the same base and height.
   Then, how do we calculate the area? We multiply the base times the height.
   But that is for two triangles.
   I want the area of only one. So I divide the product by 2.

5. ORGANIZE THE FORMULA: $A_T = bh/2$
   Show first with identifying slips "b" and "h" on the actual figure as seen in fig. 1.
   Then organize on the mat, verbalizing. . . . .$A = b \times h \div 2$
   substituting $= \dfrac{bh}{2}$

NOTE ON THE CALCULATION OF THE AREA OF THE TRIANGLE: THREE FORMULAS

We can calculate the area of the triangle with three formula expressions:

A_T:

(1) $\dfrac{bh}{2}$ (calculation of area based on the area of two triangles: thus we divide by 2)

****

(2) $\dfrac{b}{2}h$ (calculation based on half the base of the triangle times the height)

****

(3) $b\dfrac{h}{2}$ (calculation based on half the height times the base)

In our analysis of the calculation of the area of the triangle, we consider, for each kind of triangle, all three formulas. WE ALWAYS BEGIN WITH #1 in which both elements are divided by 2: a clear passage of transition from the area of the rectangle to the more complicated formulation of the last two formulas.

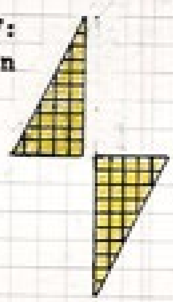
We consider each of the three kinds of triangles as classified according to angles. In the first presentation we considered the acute-angled triangle, giving formula #1. Now we proceed to formulas #2 and #3, with the acute-angled triangle.

Presentation #2: Area of the acute-angled triangle: Formula #2

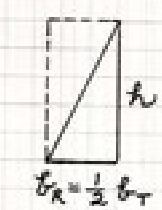
I. 1. Identify the triangle, the whole triangle as seen in fig. 1. We see that we cannot count the squares. We need a mediator.

Using the triangle divided into two right-angled triangles (fig. 2), superimpose on the first, verifying congruency. Note that the second is identical, but has been divided by one of its heights. Put the first one aside: we have a mediator.

FLIP THE RIGHT HALF: We see the only piece in the box which has the squares on the back:



MOVE THE RIGHT HALF UP TO AN ADJACENT POSITION ON THE HYPOTENUSE, THUS FORMING THE RECTANGLE:



Superimpose the real rectangle, verifying equivalence.

COMPARE THE LINES OF THE TRIANGLE AND THE RECTANGLE: the base of the rectangle is equal to 1/2 the base of the triangle. Heights are the same.

STATEMENT OF EQUIVALENCE: This rectangle is equivalent to the triangle having as its base two times the base of the rectangle and an equal height OR the triangle is equivalent to that rectangle with an equal height and a base ½ that of the △.

Presentation #2: Acute-angled triangle, formula #2. . .

II. Organizing the rule:
We note that the base of the rectangle is 1/2 the base of the triangle. We can show, then, reconstructing the whole triangle, that the base of the rectangle is 1/2 b (of the triangle.) The height remains the same.

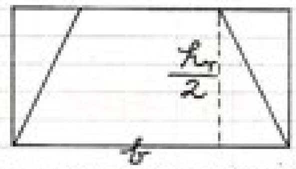
What is the area of the rectangle? b X h
What is the area, then, of the triangle?
Because the rectangle is equivalent to the triangle, we can calculate the area of the rectangle. But to express that in terms of the triangle, we see that $b_R = \frac{1}{2} b_T$.

III. Organizing the formula: SO: $A_T = \frac{b}{2} \times h$

Presentation #3: Acute-angled triangle, formula #3

I. Begin with the whole acute-angled triangle (fig. 1). Superimpose the triangle, as shown in fig. 3; that one divided by a line connecting the midpoints of the sides drawn parallel to the base, which results in the trapezoid and a triangle similar to the original one. This smaller triangle is then subsequently divided by an altitude, giving two right-angled triangles. PUT ASIDE THE WHOLE TRIANGLE: WE HAVE A MEDIATOR IN THE DIVIDED TRIANGLE.

REARRANGLE THE PIECES AS SHOWN:
Now we see that we can count the squares. SUPERIMPOSE THE REAL RECTANGLE TO VERIFY EQUIVALENCE.



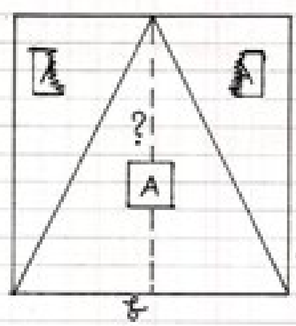
II. Organize the rule: Again we can calculate the area of the triangle in terms of the rectangle we have constructed that is equivalent. But in this case in order to express the area in terms of the triangle, we see that we have multiplied the base (of both the rectangle and the triangle) times 1/2 the HEIGHT OF THE TRIANGLE. Verify sensorially the height of the rectangle as 1/2 the height of the triangle.

III. Organize the formula: ON THE MAT: $A = b \times (h \div 2)$

$$A = b\frac{h}{2}$$

NOTE: The work of period I, in each case, is the work of the insets of equivalence repeated; and is, in fact, parallel work with that of the insets. In both we are relating all the figures to the rectangle. Therefore, we have calculated the area of the rectangle FIRST. The work of period II is also included in the early experiences here with area and provides a further understanding of the insets of equivalence.

IV. The Inverse Formulas: For all three formulas
Our consideration is primarily of formula #1: A = bh/2. If we know A and b, how do we calculate h?
NOW WE AGAIN USE THE 2, BUT IN A DIFFERENT WAY.



a) Looking at this figure, we can see that the total area is 100, the area of the triangle is 50, then the base is 10. And we can calculate the height at 10.
b) BUT we must consider that we have TWO TIMES A IN THIS FIGURE: Write a second slip for A, tear it in two, and show that we have A + ½A + ½A.
c) And because the numerical value given for A is the area of only ONE TRIANGLE, we must indicate in our formula for this inverse rule the division of the double of the area by 2 in order to calculate the area of only one triangle.
d) SHOW ON THE MAT BOTH SLIPS OF A, one on top of the other: then replace with the "2" A. And complete the formula.
e) Remove the 2 from the formula and see the incorrect result:

$\overset{?}{h} = \boxed{A} \div b$

$h = 2A \div b$

$h = \frac{2A}{b}$

Repeat the experience with the unknown element as b. NOTE that it is only necessary to switch the positions of b and h.

$$b = \frac{2A}{h}$$

Presentation #4: **Right-Angled Triangle: Formula #1**

Material:

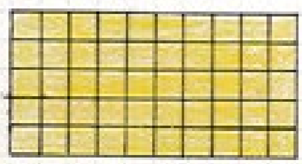


fig. 4

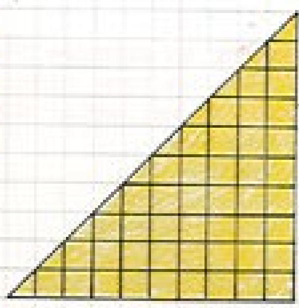


fig 5.

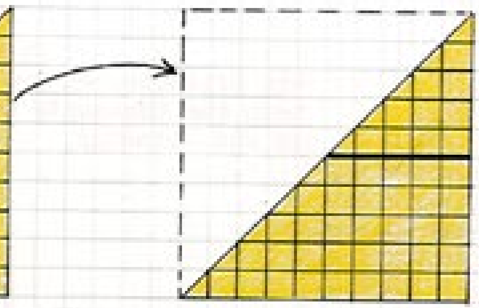


fig. 6

The rectangle: for reference.

The whole right-angled triangle.

The right-angled triangle formed of two pieces; that right-angled triangle being divided by an axis drawn from one of the legs perpendicular to the hypotenuse.

I. Begin with the whole right-angled triangle: IDENTIFICATION. We see that we cannot count the squares: we have only HALF A SQUARE. Take the divided right-angled triangle as shown in fig. 6; SUPERIMPOSE TO VERIFY CONGRUENCE. Then form a square with the two, as indicated in fig. 6.

We have doubled the first figure: which is consistent with the formulation of our first formula.

Now we can count the squares.

II. **Organize the rule:** Show labels on "b" and "h" on the figure. EMPHASIZE THAT WE MUST DIVIDE THE PRODUCT BY 2 or we will have the area of two triangles.

III. Organize the formula:

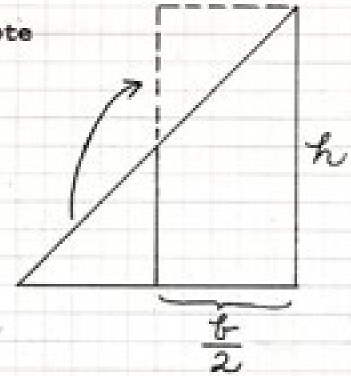
$$A = (b \times h) \div 2$$

$$A = \frac{bh}{2}$$

Presentation #5: **Right-Angled Triangle: Formula #2** Material the same.

I. <u>Identify</u> the right-angled triangle, using that one divided (fig. 6). We cannot count the squares. VERIFY CONGRUENCY WITH THE WHOLE TRIANGLE. Note how the mediating triangle has been divided:

The Yellow Material of Area. . .
Right-Angled Triangle: Formula #2. . .

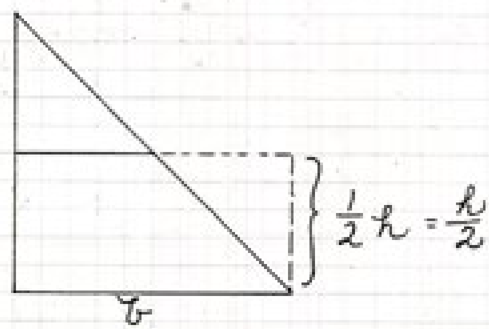


SHOW THE DIVIDED TRIANGLE IN THIS POSITION: Note that the triangle has been divided by drawing a perpendicular from the mid-point of the base (axis). . . This indicates that the base has been divided into two equal parts.
Transform the triangle into the rectangle.
Show the real rectangle superimposed on the one formed: STATE CONGRUENCE..
SHOW THE WHOLE TRIANGLE: State EQUIVALENCE:
Compare the lines of the rectangle formed and those of the whole triangle, identifying with slips "b" and "h" of the rectangle as shown.

II. Organize the rule: We have identified the base of the rectangle as 1/2 the total base of the triangle. (VERIFY SENSORIALLY) So in order to calculate the area of the triangle which is equivalent to this rectangle, we multiply the "h" (same for both figures) times $\frac{1}{2}$ the base of the triangle (which is the base of the rectangle.)

III. Organize the formula: $A = \frac{b}{2} h$

Presentation #6: Right-angled triangle: Formula #3



I. Show the divided right-angled triangle now in this position. IDENTIFY: Now the height is divided into two equal parts. (Show this sensorially)
VERIFY CONGRUENCE WITH THE WHOLE TRIANGLE.
TRANSFORM INTO THE RECTANGLE AS SHOWN.
VERIFY CONGRUENCE WITH THE WHOLE RECTANGLE.
STATE EQUIVALENCE.
Now we can count the squares.

II. Organize the rule: The base of the triangle corresponds to the base of the rectangle. But we see that the height of the rectangle is equal to $\frac{1}{2}$ the height of the triangle. So, to calculate the area of the triangle (which is equivalent to the rectangle), we multiply the base times one-half the height.

III. Organize the formula: $A = \frac{h}{2} b$

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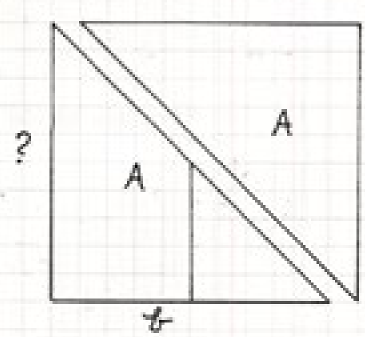
IV. Inverse formula: for $A = \frac{bh}{2}$



a) Show the square composed of the two right-angled triangles: identify each with A.
b) Problem: If we know A of the triangle, and either b or h; looking for the other element.
c) EMPHASIZE: When we know the area of the triangle, we have two times that area when looking for "b" or "h". I MUST DIVIDE THE DOUBLE OF THE AREA BY THE ELEMENT WHICH I KNOW IN ORDER TO FIND THE OTHER ELEMENT. We must have that 2A because we have two figures on which we base our search for the base or height.
d) With slips on the mat, organize the formula, showing the elements first on the figure, then utilizing the 2 to indicate two times the area.

$$\overset{?}{h} = \frac{2A}{b}$$

OR

$$\overset{?}{b} = \frac{2A}{h}$$

Presentation #7: Obtuse-Angled Triangle: Formula #1

Materials: As shown in fig. 7: One whole obtuse-angled triangle; a second one constructed of three parts.

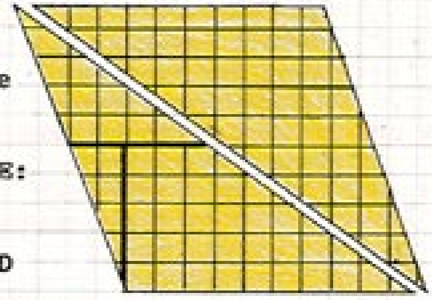


fig. 7

I. IDENTIFY THE WHOLE OBTUSE-ANGLED TRIANGLE: We cannot count the squares.
VERIFY CONGRUENCY BETWEEN THE WHOLE AND THE OBTUSE-ANGLED TRIANGLE FORMED OF THREE PARTS, superimposing.
Show the two together: identify the resulting common parallelogram.

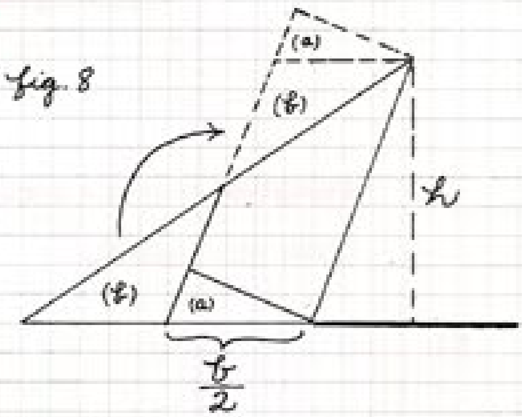
II. Organize the rule: We know how the calculate the area of the common parallelogram: $A_P = bh$ . . .BUT we don't want the area of that common parallelogram. We want the area of only one triangle which we know is one-half the figure. So we divide the formula by 2.

III. Organize the formula: $A_T = (b \times h) \quad 2$ OR $A_T = \frac{bh}{2}$

Presentation #8: Obtuse-angled Triangle: Formula #2

I. VERIFY CONGRUENCY BETWEEN THE WHOLE OBTUSE-ANGLED TRIANGLE AND THE DIVIDED TRIANGLE.
Show that divided triangle as in fig. 8.
IDENTIFY BASE AND HEIGHT: we must prolong the base in order to determine the external altitude. (NOTE: here only the height is NOT 10)
Transform to the common parallelogram.
We can continue and transform this parallelogram into the rectangle, but it is not necessary because we KNOW HOW TO CALCULATE THE AREA OF THE PARALLELOGRAM.

fig. 8



II. Organize the rule: We can calculate the area of the common parallelogram by multiplying the base times the height. We know that the triangle is EQUIVALENT to this common parallelogram. (VERIFY by showing the transformation.) To express the area formula for the triangle we have the same height, but we see that the base of the triangle is only one-half that of the parallelogram. (NOTE the median drawn from the base)

III. Organize the formula: SO we calculate the area of the triangle: $A = \frac{b}{2} \times h$ OR $A = \frac{b}{2} h$

Presentation #9: Obtuse-Angled Triangle: Formula #3

I. Here we change the position of the triangle noting now that line parallel to the base which is drawn from the mid-point of the side to the mid-point of the adjacent side: we have divided the height into equal (2) parts. The result is an obtuse-angled trapezoid and a triangle similar to the original one.
VERIFY THE EQUAL HEIGHTS OF THE SMALL TRIANGLE AND THE TRAPEZOID SENSORIALLY.
Transform the triangle into the common parallelogram as shown in fig. 9.

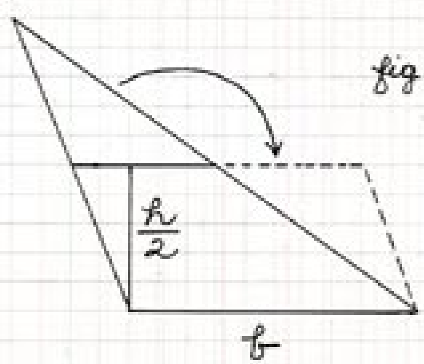


fig. 9

II. Organize the rule: We can calculate the area of this common parallelogram by multiplying the base times the height. We have shown that the height of the triangle is one-half the height of the rectangle. Therefore, we can calculate the area of the triangle which is equivalent to this common parallelogram by multiplying base times $\frac{1}{2}$ height.

III. Organize the formula: $A = b \times \frac{h}{2}$ OR $A = b \frac{h}{2}$

ACTIVITIES: Invite the child to construct the figures with squared paper, using different measures than those of the material. (We are looking still for the number of squares: not the size of the square which is a problem of measure, a later work)

AGES: Period I (of all the work): Parallel to the Insets of Equivalence
Period II: 8
Period III: 9
Period IV: 10

DIRECT AIMS:
1. The calculation of the area of some plane figures.
2. Concept that the square is the measure of area. (A conjunctive concept with that of the triangle as the constructor.)

INDIRECT AIM: Preparation for the calculation of lateral and total surface of solids.

****

AREA OF THE SQUARE

Material
1. From the plane insets, the whole square.
2. The rectangle from the yellow material.
3. White labels: b | h | ÷ | = | ? | X | 2 | A

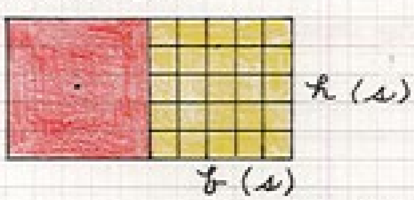
Presentation

I. Identification and Equivalence:

1. Showing the square from the plane insets, identify the base and height of the square.

   1. When this side is the base, this side is the height. . .we can show another base, another height.

2. Superimpose the metal square inset on the yellow rectangle.



   2. By counting the squares of the yellow figure which is now shown, we see that we have a square: it is the same square that we have in the inset, but this one is divided into little squares.

3. Count the squares: analyze how the number of squares was obtained. TURN OVER THE SLIPS SHOWN ON THE FIGURE WHICH READ "b" and "h" and write on both "s"

   3. There are 25 squares in our square. How was that 25 obtained? 5 X 5 What is 5? the base. AND the height. BOTH DIMENSIONS ARE THE SAME: The base equals the height. We will call both "side"---"s"

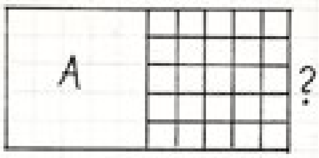
II. Organize the rule: To calculate the area of the square, we multiply one side times another side: s X s.

III. Organize the formula: $A = s \times s$ OR $A = s^2$

NOTE: The "2" we used as a denominator we will now use as an exponent: the technique of fractions at the arithmetic level.

IV. Inverse Rule: an interesting inverse rule because we find that our rule is the square root of the area (A).

The Problem:



If I give you the square of 100 beads, what is the side? 10
If I give you the square of 64 beads, what is the side? 8
If I give you the square of 25 beads, what is the side? 5
Here the beads are squares.

Now---from the number of squares, we are looking for the side of the square.

REVERSE THE FORMULA: $s^2 = A$ SHOW THE INVERSE: $? = A$

REMOVE THE EXPONENT---PREPARE THE SQUARE ROOT SIGN AND SHOW:

$$s = \sqrt[2]{A}$$

And, in fact $\sqrt[2]{25} = 5$

If I remove the exponent, we have divided the side by itself.
When we remove the exponent, we have divided "s" by itself; we have carried out the square root of "s" SO. . .we must carry out the square root of A.
The square root of A will be "s."

THE AREA OF THE RHOMBUS

The rhombus is a special parallelogram with four equal sides. We can calculate the area of the rhombus in four ways:

1) $A_{rh} = bh$ Because the rhombus is a special case of the parallelogram, we know that we can calculate the area of the rhombus as we would calculate the area of the parallelogram. WE RECALL THIS PASSAGE FROM THE INSETS OF EQUIVALENCE.

2) $A_{rh} = \frac{Dd}{2}$ D is the long diagonal of the rhombus; d is the short diagonal. The area can be calculated as the semi-product of the two.

3) $A_{rh} = \frac{D}{2}d$ Corollary #1 of the second formula. As in the resulting formulas #2 and #3 of the triangle area.

4) $A_{rh} = D\frac{d}{2}$ Corollary #2

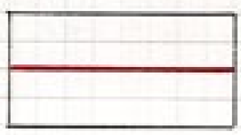
Because we have already discovered the area calculation for the parallelogram, we begin our investigation of the calculation of the area of the rhombus with formula #2.

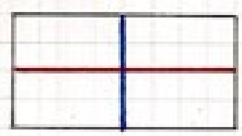
Material

1. A sheet of paper, rectangular.
2. Ruler, scissors, three pens of different colors. (Wide tips)
3. From the INSETS OF EQUIVALENCE: #2, #4, #17

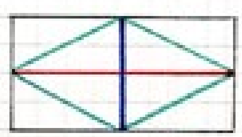
Presentation #1: Formula #2 for the Area of the Rhombus: Paper Construction

1. Fold the rectangular piece of paper along the longest side: open and trace the fold with a colored pen---a wide line that goes on both sides of the fold.



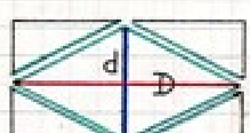


2. Fold the paper along the short side, and trace the fold with a second color.



3. Connect the end points of the two lines in a third color.

4. Cut out the rhombus so that the colored line shows on both pieces of each cut.



5. Reconstruct the rectangle, showing the two possible orientations of the rhombus:

6. Identify with labels D and d: I am only interested in the diagonals for this work.

Area of the Rhombus. . .
Presentation #1: Formula #2: Paper Construction

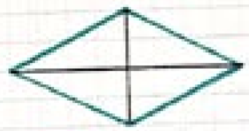
7. IDENTIFY THE BASE OF THE PARALLELOGRAM AS "D" and the HEIGHT AS "d" :
Switch the labels to that position.

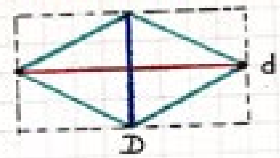
II. Organize the rule: Because the long diagonal (D) of the rhombus is equal to the base of the parallelogram, and the short diagonal (d) is equal to the height of the parallelogram, we can calculate the area of the rhombus as $A = D \times d$.

BUT

We have to show that product divided by 2 or we will have the area of two rhombi.

SHOW THE SECOND RHOMBUS, first taking the corner pieces and verifying congruency with the first rhombus; and then forming the second rhombus:



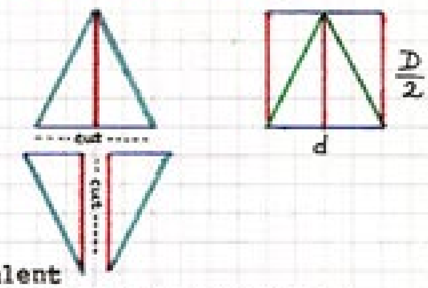


III. Organize the formula: $A = \frac{D \times d}{2}$

Presentation #2: Formula #3: First Corollary for the Area of the Rhombus: Paper

I. 1. Take the whole rhombus as constructed in presentation #1 and make two cuts:

2. Join the two small triangles to the large one along the green lines, thus transforming the figure to a rectangle. Identify $\frac{D}{2}$ and d.



II. Organize the rule: Now we can calculate the area of that rectangle which we see is equivalent to the rhombus. The height (h) of the rectangle is equal to the long diagonal of the rhombus divided by 2 and the base is equal to the short diagonal.

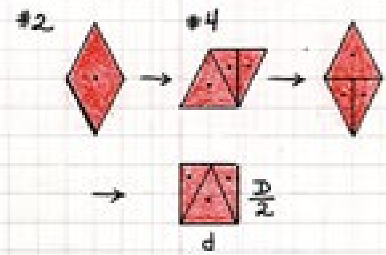
III. Organize the formula: $A = d \times \frac{D}{2}$

Presentation #3: Formula #3: First Corollary: with the Insets of Equivalence

1. Show frame #2: IDENTIFY D and d.

2. Substitute in the frame the divided rhombus of frame #4, VERIFYING CONGRUENCE.

3. IDENTIFY the short diagonal (d) and one-half the long diagonal (D/2).

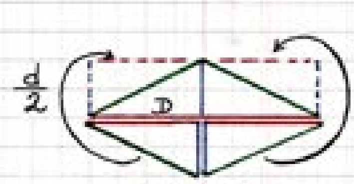


4. Repeat the paper conversion to the rectangle.

5. We have a parallelogram whose dimensions are $\frac{D}{2} \times d$.

Presentation #4: Formula #4: Second Corollary for Area of Rhombus: Paper Construction

I. With the three pieces used in the presentation of the first corollary, reconstruct the rhombus.
CUT the remaining half of the long diagonal.
JOIN TWO SMALL TRIANGLES ON THE HALF SHORT DIAGONAL AS SHOWN, forming a long triangle.
Transform the figure into a rectangle.



II. Organize the rule: The height of the rectangle is equal to one-half the short diagonal (d/2) and the base is the long diagonal (D). So we can calculate the area of the rectangle (and rhombus):

III. Organize the formula: $A = D \times \frac{d}{2}$

Presentation #5: Formula #4: Second Corollary: With the Insets of Equivalence

1. Orient frame #2 so that the long diagonal is parallel to you: Put frame #4 aside.

2. From the third figure of frame #17, take the divided rhombus and substitute in frame #2, verifying congruency.

3. Substitute the two right-angled triangles from frame #4 for one of the two obtuse-angled triangles of frame #17, now shown in #2.

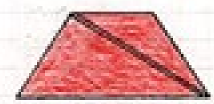
4. IDENTIFY THE LONG DIAGONAL (D) AND ONE-HALF THE SHORT DIAGONAL (d). AND TRANSFORM THE FIGURE INTO THE RECTANGLE.

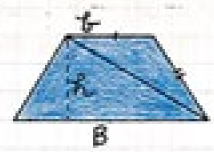
5. Restate the corollary and the formula: $A = D \times \frac{d}{2}$

THE AREA OF THE TRAPEZOID

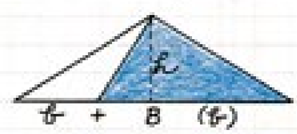
The area of the trapezoid can be calculated in three different ways:

1) As seen in the constructive triangles, Series #1: Box #1 (red pair of triangles form the trapezoid) and Box #2 (blue triangles). Both pairs of triangles as found in these boxes construct an **isosceles trapezoid.** The component parts of that trapezoid are the right-angled scalene triangle and the obtuse-angled triangle (which is better if it is not the special case of the isosceles obtuse-angled triangle: depends on the particular set of the materials.)





In our first work with this trapezoid, we simply unite the two triangles to form the figure. In the second work, the second box, we transform that figure into the triangle as shown:
NOTE: without the isosceles obtuse-angled triangle, it is necessary to turn over that triangle in this construction.



$A_T = \frac{bh}{2}$

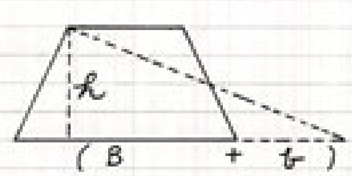
$A_{tr} = \frac{(B+b)h}{2}$

Because we know the calculation for the area of the triangle, we can now calculate the area of the equivalent trapezoid as shown, using the expression (B + b) to indicate the position of the major and minor bases of the trapezoid as the base.

2) The passage directly from the trapezoid to the rectangle, without the intermediate passage to the triangle, as seen in the work of frame #10 from the insets of equivalence. In this work we conclude that the trapezoid is equivalent to that rectangle having a base equal to the sum of the major and minor bases of the trapezoid, and a height equal to ½ the height of the trapezoid. With this information, then, we can calculate the area of the trapezoid according to the formula for the area of the rectangle:

$A_R = bh \qquad A_{tr} = (B + b)\frac{h}{2}$

3) The third mode of calculating the area of the trapezoid is preferred by mathematicians. In it we go directly from the trapezoid to the triangle as shown: and that completes the calculation of the area of the trapezoid. IT IS THIS THIRD METHOD WHICH WE WILL UTILIZE TO SHOW THE CALCULATION OF THE AREA OF THE TRAPEZOID.



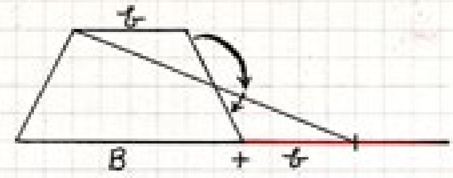
$A_T = \frac{bh}{2}$

$A_{tr} = \frac{(B+b)h}{2}$

The Area of the Trapezoid. . .

Presentation: Area of the Trapezoid

I. 1. From the drawing of the trapezoid, we first PROLONG THE BASE.



2. On that base, we mark a segment which is as long as the minor base (b).

3. Unite the vertex to that point on the prolonged base.

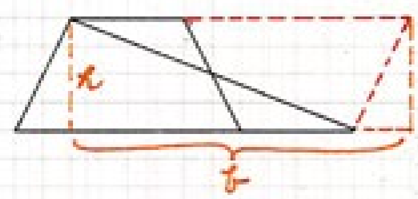
3. We have divided the oblique side of the trapezoid into two equal parts.

4. USING A PAPER FIGURE OF THE TRAPEZOID NOW, MAKE THE SENSORIAL PROOF, folding slightly to indicate the equal division of the oblique sides, then drawing a line from the vertex to that fold. CUT. Then pivot that triangle from the upper part of the trapezoid on the vertex: one-half of the oblique side will coincide equally with the other and the minor base (b) will coincide with the marked segment on the base.

II. 5. Organize the rule: We have formed a triangle, the base of which is (B + b) and the height of which is equal to the height of the trapezoid. So, knowing that this triangle is equivalent to the trapezoid, we can calculate the area of the trapezoid:

III. 6. Organize the formula: $A_{tr} = \frac{(B + b) \times h}{2}$

NOTE: We know that we must divide the product of (B + b) X h by 2 because that is our formula for the area of the triangle: $A_T = \frac{bh}{2}$



BUT WE ALSO CAN OBSERVE that we must divide this product by 2 or we will have the area for the corresponding parallelogram of the figure OR the area of TWO trapezoids. FINALLY WE SEE THAT WE COULD TRANSFORM THE FIGURE INTO A RECTANGLE: the rectangle is our arriving point.

THE AREA OF POLYGONS

The polygon may be:

A. Concave: As seen in the Constructive Triangles, Series #1, Box #2, transforming the two triangles which form the trapezoid into a concave quadrilateral.
As seen in the Constructive Triangles, Series #2, Box H_1, with the formation of the "arrowhead."

B. Convex:

1. Irregular: We have met the irregular polygons in the work of the classified nomenclature. We have considered the area of the common quadrilateral as a transformation to triangles, noting that we are unable to calculate that area as we would a rectangle for lack of a base and altitude.
2. Regular: Again we have met the regular polygons through the work of the classified nomenclature and in the geometry cabinet. We have considered the calculation of area specifically for:
    - a) The equilateral triangle: The consideration of the area of the triangle includes this special case.
    - b) The square: A consideration first of the calculation of area as a special case of the rectangle; and then as the regular polygon (the only square) in the group of quadrilaterals.
    - c) The pentagon: In the work of the insets of equivalence, we have approached the area calculation for the regular pentagon in the group of pentagons. . .and ONLY the regular pentagons.
    - d) The decagon: In the work of the insets of equivalence, we have considered the area of the decagon, ONLY the regular decagon in the group of decagons.

    WE HAVE NOT CONSIDERED THE CALCULATION OF AREA FOR:

    - e) Those regular polygons which fall between the pentagon and the decagon.
    - f) Nor those which take us beyond the decagon to "n" number of sides. . .
    - g) Nor the limit of the regular polygons. . .the circle.

The Area of Polygons. . .

Now we examine the area of regular polygons. In the series of regular convex polygons, we include the equilateral triangle and the square. We know the area calculation for these two: we have considered the square as a particular case of the parallelogram; and the equilateral triangle as a special case of the triangle. Now we include these two becuase the formula we have constructed in the end is not the formula used by mathematicians to calculate the area of any regular polygon.

Mathematicians give a formula that is valid for all regular polygons:

To calculate the area of regular polygons: A_{rp}

p = perimeter
a = apothem

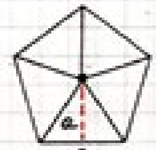
The general formula: $A_{rp} = \frac{pa}{2}$

Corollary #1: $A_{rp} = \frac{p}{2} a$

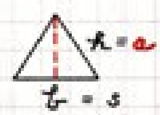
Corollary #2: $A_{rp} = p \frac{a}{2}$

In Montessori's approach to regular polygons with more than four sides, we have met the solution, through the insets of equivalence, for the pentagon and the decagon. In the Montessori solution, we move directly from the decagon to the rectangles without first going through the triangle. At the level of equivalence, we can make this passage. Just as for the trapezoid we move to the rectangle in order to do the work of counting the squares. But for the geometrical demonstration we must go from the polygon to the triangle.

If we show a center in the regular polygon, we imply that we can circumscribe a circle with that center around the figure; and that we can inscribe a circle with that center within. So the apothem is the radius of the inscribed circle. The center represents the center of symmetry.



When we unite the center to all the vertices of the regular polygon, we divide it into triangles which are equal among themselves. The lines creating these divisions represent the radii of the circumscribed circle.



So our solution, with the regular pehtagon is the give equal triangles which we have created. If we calculate the area of one of those triangles and multiply that area times the number of sides of the regular polygon, we have the area of the whole figure. Thus we find the area of any regular polygon. In our solution, the problem goes to the one section which is in turn equal to all the other sections we have made in the figure. We have a triangle whose base is equal to the side of the regular polygon and whose altitude is equal to the apothem.

Theory of Constants: In this solution, we need to know only "s" Through certain passages we can find the area of the polygon from only this dimension. This implies that there is a way, from the side, to determine the apothem.

In all regular polygons (except one) there is a constant irrational number which allows us to find the area, knowing only its side.

Presentation #1: Area of the Regular Decagon

Material: Insets of Equivalence: frames #13, #15, #16

1. SHOW THE DECAGON FROM #13 on the mat in a horizontal position with the two rectangles of frames #15 and #16.

1. Our conclusion from the insets of equivalence: This decagon is equivalent to both rectangles.

2. Using the decagon and the rectangle from #15, organize the nomenclature for Corollary #1 for the area of the regular polygon: USE SMALL LABLES

$A_D = \frac{p}{2} a$

2. The decagon is equivalent to that rectangle having a base equal to $\frac{1}{2}$ the perimeter of the decagon and a height equal to the apothem of the decagon.

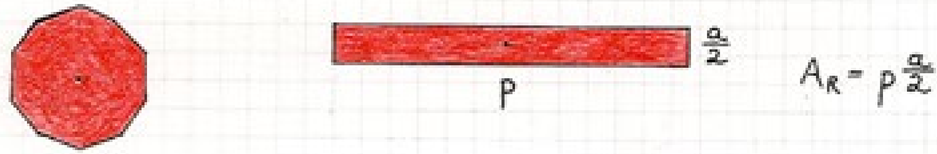
$A_R = bh = \frac{p}{2} a$

The Area of Regular Polygons. . .
Regular Decagon. . .

3. Using the decagon and the rectangle from frame #16, organize the nomenclature for Corollary #2:

$$A_D = p\frac{a}{2}$$

3. The decagon is equivalent to that rectangle having a base equal to the perimeter of the decagon and a height equal to one-half the apothem of the decagon.



Presentation #2: The Study of the Apothem

Material
1. The drawer of regular polygons from the geometry cabinet.
2. The circle metal inset with the triangle inscribed.
3. The square metal inset showing the square divided into fourths by the diagonals.
4. From the drawer of circles of the geometry cabinet: the largest circle.

1. DISPLAY Horizontally on the mat:

The Triangle (taken from the center of the inset: this one chosen because it is inscribed in a circle of 10 cm. diameter; the equilateral triangle from the plane insets has a side of 10 cm.)

The Square (removing two fourths from the metal inset frame to form a square that is inscribed in a circle with diameter of 10 cm.)

The Pentagon
The Hexagon
The Heptagon
The Octagon
The Nonagon
The Decagon
THE CIRCLE

SO. . .all the figures can be inscribed in the same circle of diameter 10 cm.

2. Using the circle frame (of 10 cm. diameter) demonstrate that all the figures can be inscribed in the same circle.

3. Commentary on the apothem: We have seen that each regular polygon can be divided in equal parts. The first figure, being the constructor (the triangle) can only be divided into smaller triangles. All the other figures can be divided into equal triangles.

If we want to divide the circle into triangles, the base must be a point and the oblique sides two rays (radii) which are superimposed above that point.

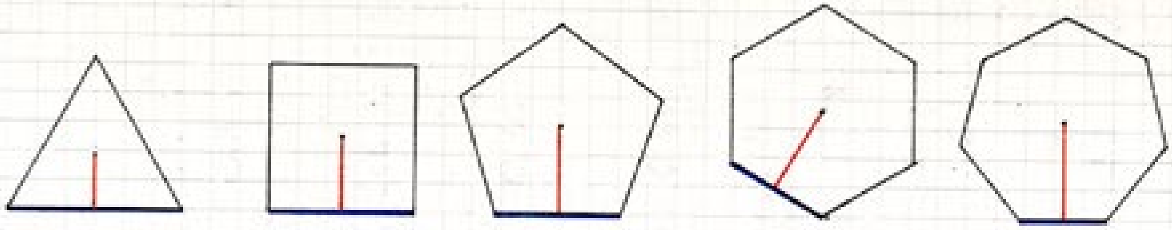
I have discovered that these are all regular polygons. And that they all have an apothem: which is the radius of the inscribed circle. Even the triangle and the square have an apothem.

DEFINE APOTHEM: I can inscribe a circle in the triangle (shown) and the little knob represents the center of that circle, corresponding to the center of the first regular polygon of reality. The line uniting that center and the midpoint of any side of the triangle is the apothem.

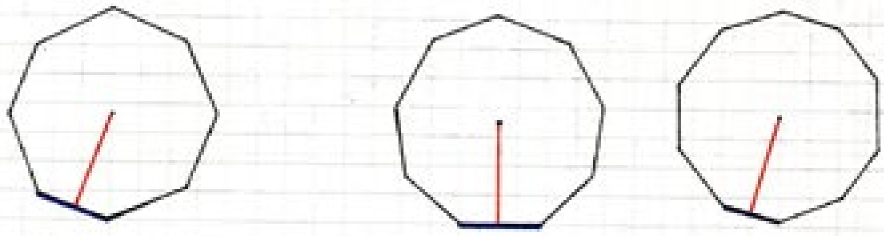
Does the square have an apothem? Yes. Uniting opposite vertices we have a center. The center of the square coincides with the center of the circle. Joining that center with the mid-point of the sides of the square gives us the apothem (s).

4. The child traces the perimeter of each figure and draws the apothem in red. Then he identifies the side functioning as base by marking it in blue. This is the beginning of the RESEARCH OF THE CONSTANTS WHICH WILL BE THE RATIO BETWEEN THE RED LINE AND THE BLUE, between the length of the side and the apothem.
   a) The child compares the lines of two different colors, first sensorially with his eyes, judging which line in each figure is longer.
   b) He may write this evaluation, naming the lines "apothem" and "side."
      OR "The blue line is shorter than the red line."
   c) At a certain point it becomes difficult to make this judgment: the child uses a ruler: He examines all the regular polygons from the triangle to the decagon in this way.
   d) The child discovers that: the apothem is either shorter than or longer than the side, but never equal to the length of the side.
   e) NOW HE WRITES ON EACH FIGURE: $a < s$ OR $s > a$

Area of the Regular Polygons. . .
Study of the Apothem. . .



$s > a$ / $a < s$ | $s > a$ / $a < s$ | $s > a$ / $a < s$ | $s > a$ / $a < s$ | $s < a$ / $a > s$



$s < a$ / $a > s$ | $s < a$ / $a > s$ | $s < a$ / $a > s$

f) We examine the statements the child has made; now he writes the relationship again, with the first term constant. (Above: first term is always s) AND second, the first term is always "a." *****

g) We discover that: in the first four regular polygons of reality the side is greater than the apothem. All those regular polygons which follow the hexagon show the opposite relationship: the side is less than the apothem.

*****For our next work, we MUST ESTABLISH "a" AS THE FIRST TERM.

Presentation #2: Discovering the First Digit of the Constant: Sensorial Level

With the child, we make the following observations based on his previous work with the comparison of the "red and blue lines" in the regular polygons.

1) We have discovered that: A. $a \lesseqgtr s \longrightarrow a \neq s$
B. $s \lesseqgtr a \longrightarrow s \neq a$

2) We must establish "a" as our first term because it is the apothem which we want to know from the dimension of the side.

3) We have seen that from the triangle to the hexagon the apothem is less than the side; and from the heptagon to the decagon to that regular polygon of an infinite number of sides, the apothem is greater than the side.

$\triangle \longrightarrow$ hexagon	heptagon $\longrightarrow \infty$
$a < s$	$a > s$
If $s = 1$ (unit)	
$a < 1$	$a > 1$
Then the first digits:	
0._____	1._____ (or 2, or 3; or x.)_____) because the apothem continues to increase as number of sides increase

The ... the apothem ... studio dell' apotema.

Presentation #3: Discovering the First Decimal Digit of the Constant: The Ratio Between the Apothem and the Side

We know that the number we write will be the result of. . . . . $a \div s = \frac{a}{s}$

1. We begin with an examination of the square, the only regular polygon, the constant for which will be a rational number.
   We have already established that:
   if s = 1
   and a < s (square)
   then a < 1

   The Square: How many times is the blue line contained in the red? ½ time

   So the constant for the square is: 0.5

2. In the same way, with the children we consider the ration between the red line and the blue line in each of the figures we have examined.
   Our consideration now is: How many times the blue line is contained in the red? And we have certain guide lines to follow according to the information which we have gathered so far.
   So for the pentagon: $\frac{a}{s}$ $a < 1$ $k = 0.6 \ldots$ cir.

NOTE: In discovering the first decimal digit, we find that all other figures give an irrational number as the constant. We emphasize this imperfect resolution of the number, writing (. . .cir.) A chart showing the comparison of the red and blue lines, marked off in units to indicate "number of times contained" is a help in graphically demonstrating the formation of the constant.

3. Give the children the specific constants, as formulated "by the book."

Equilateral triangle	0.288. . .cir.
Square	0.5. .
Pentagon	0.688 . . cir.
Hexagon	0.866 . . cir.
Heptagon	1.039 . . cir.
Octagon	1.207 . . cir.
Nonagon	1.373 . .cir.
Decagon	1.538 . . cir.
Eleven sides	1.702. . cir.
Dodecagon	1.866 . .cir.
Fifteen sides	2.352. . cir.
Twenty sides	3.151 . .cir.

4. Examining the list, we observe:
   The increase in the whole number of the constant as the number of sides increase.
   We also know that the apothem increases with the number of sides increase.
   And that the length of the side decreases with the increase in number of sides.

Concluding: The length of the apothem progressively increases with the number of sides: it begins smaller than the sides, then gradually becomes longer than it because the polygons examined are all inscribed in the same circle of diameter 10 cm.

Presentation #4: Does the Constant Change?

We know now that there is a constant for each regular polygon. The question at this point is whether or not that constant changes.

To solve this dilemma, Montessori suggests: that the children construct the regular polygons from the triangle to the decagon in three different measures: with a side of 1 cm., a side of 10 cm., and a side of 100 cm.

In this work, then, the child will have 24 figures, representing eight groups of three similar figures each.

He discovers that: the constants are valid for all similar figures.

Area of Regular Polygons

Presentation #5: The Formula Utilizing the Constants

$$A_{RegPol} = \frac{p \times a}{2}$$

p = n (number of sides) × s

a = s × k (constant)

Then $A_{RP} = \frac{(s \times n) \cdot (s \times k)}{2}$

$= \frac{s^2 \times n \times k}{2}$

$= s^2 \times n \times \frac{k}{2}$

The rule: The area of any regular polygon can be calculated by multiplying the square of the side (s^2) times the number of sides times the constant of the corresponding polygon.

And we can use this formula at a certain point with the information we have gathered.

$\triangle = s^2 \times 3 \times \frac{0.28}{2}$

$= s^2 \times 3 \times 0.14$

$= s^2 \times 0.42$

Area of Regular Polygons: The Triangle Solution

1. We can show a regular polygon, such as the pentagon, divided first by two lines from the center to adjacent vertices, indicating the proposed division into triangles. Then we show and identify also the altitude of that triangle in green (the apothem.)

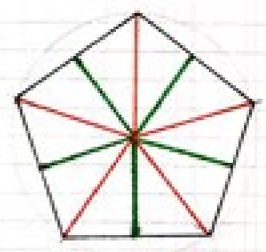


fig. 1

2. We further divide that polygon (pentagon) as shown in fig. 1.

3. In fig. 2 we shown the triangles which compose our polygon on a long base representing the perimeter. NOTE: The polygon (regular) divided into triangles will always result in a series of isosceles triangles, the exception being the hexagon where we have the equilateral triangle, a special case of the isosceles.

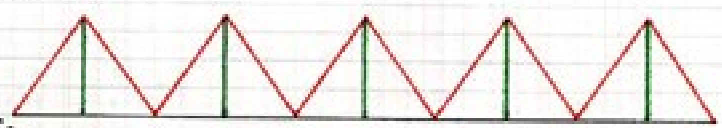


fig. 2

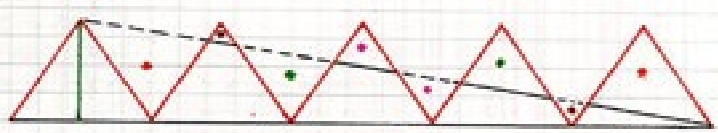


fig. 3

4. In fig. 3, then, we transform the triangles into ONE TRIANGLE by drawing a line from ANY VERTEX to the end point(s) of the perimeter. (When we begin at the center apex, the result is the isosceles triangle, possible only with the polygon of odd-numbered sides.) Here the result is a scalene triangle. (fig. 4)

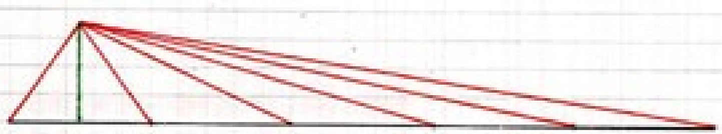


fig. 5

5. In figure 4, we see that triangle which gives us the dimensions for the calculation of the area of the polygon: $A = \frac{pa}{2}$

6. In fig. 5, we see the actual triangles into which the first five have been transformed. Each is a triangle equivalent to those first five which were congruent among themselves because all have equal base and equal altitude (apothem.)

THE AREA OF THE CIRCLE

Presentation #1: MEASURING THE CIRCUMFERENCE: The Circle Understood as the Limit of the Regular Polygons

AIM: A sensorial identification of the circle as a regular polygon.

Material

1. From the geometry cabinet: the drawer of the circles and the drawer of regular polygons.
2. From the metal insets: the triangle inscribed in the circle (equilateral); and the square divided into fourths by diagonals.
3. Wooden board, chalk, ruler.

1. Show on the mat the frame of the largest circle (from the drawer of circles.)

2. Take first the inscribed triangle, putting aside the frame and other parts; and show the triangle in the frame of the 10 cm. circle.

2. When we inscribe the triangle in the circle, we see three big spaces uncovered.

3. Repeat the experience with the square (two fourths from the metal inset, shown as a square with a diagonal of 10 cm.), then the pentagon and all the other regular polygons, noting that with the square the spaces are smaller and there are four; with the pentagon, the spaces are even smaller and there are five. . . Then, with the decagon, the spaces are very small and there are ten.

4. Imagine a regular polygon of twenty sides. . .of 100 sides. . .of 1,000 sides: the spaces would always become smaller and the number of them would increase.

5. Show the circle in the frame, using the 10 cm. circle. It fits exactly.
NOTE: We now have a visual picture of the circle as a negative and positive of the same image.

5. The number of sides in our regular polygons could increase to infinity. . . until we come to the circle.
How many sides are there?
They cannot be counted.
The circle is a regular polygon of an infinite number of sides. Each point of the circumference is a side.

Presentation #2: MEASURING THE CIRCUMFERENCE: Transfer of the Nomenclature of a Regular Polygon to the Circle

1. Show, from the geometry cabinet drawers, the plane figures of the DECAGON and CIRCLE.

2. Give the nomenclature of both simultaneously, noting the corresponding terminology for the corresponding parts:

	THE DECAGON	THE CIRCLE
A.	The surface of the decagon is limited by a number of sides.	The surface of the circle is limited by a curved closed line composed of many points.
B.	The sides together form the perimeter.	The series of points form the circumference of the circle.
C.	The regular polygon has a center. . . which is equidistant from all vertices.	The circle has a center. . .which is equidistant from every point on the circumference.
D.	The line segment drawn from the center to the mid-point of the side of the regular polygon is called the apothem.	The line drawn from the center of the circle to any point on the circumference is called the radius.

3. Conclusion: For both of these regular polygons, the decagon and the circle, we have the same lines, but we give them different names.