CHAPTER THREE: of the GOLDEN CHAPTER OF GEOMETRY FOR THE ELEMENTARY SCHOOL: THE INSETS OF EQUIVALENT FIGURES

The study we have made with the constructive triangles has opened two roads to us:

1) The study of the relationship of lines in equivalent figures, one which we have begun sensorially in the constructive triangles. Here we study these relationships in detail.
2) The study of the areas; the direct application of the study of the relationship of lines.

With the insets of equivalent figures, we begin the first study: that of the relationship of lines in equivalent figures. The materials for this study are the insets (some plastic, the majority metal) of equivalence. There are 20 insets in all. And they are grouped in families for our study. The first six families, A - F, provide a comparison study in which the first term of the comparison varies from family to family, but in each of which the second term is the rectangle. The last family G are three insets of the Pythagorean Theorem, the classical application of equivalence. Here again, we see the equivalence relationship in terms of the rectangle. We note now the basic composition of the seven families:

A) Equivalence between the triangle and the rectangle: inset frame #1.

B) Equivalence between the rhombus and the rectangle: insets #2, #3, #4.

C) Equivalence between the common parallelogram and the rectangle: #5.

D) Equivalence between the trapezoid and the rectangle: #6, #7, #8, #9, #10.

E) Equivalence between the pentagon and the decagon (#11, #12, #13, #14) and the rectangle. (#15, #16.) Regular polygons and the rectangle.

F) The Theorem of the Application of Equivalence: #17.

G) The Pythagorean Theorem: #18, #19, #20.

REFERENCE: The Montessori Elementary Material, Vol. II; pp. 277 - 291

NOTE: In the chapter of geometry in the volumn mentioned above, Dott.sa Montessori unites the two experiences: of equivalence and area. This book, The Montessori Elementary Material, Vol.II, was first published in 1916. In 1934 Psycho-Geometry was published. From 1934 until her death in 1952, Montessori continued to work on the subject.

In 1916 she was referring to the material of equivalence for the discussion of area, because the materials for area did not yet exist. Then, in Psycho-Geometry, 1934, she talked for the first time about the area material, making a clear distinction between the materials of equivalence and those of area. Thus our work with the insets of equivalence deal solely with that relationship; and with further work we consider area.

The general aim of the work done with equivalence is to make the child aware of the reason why certain figures are equivalent. The child has worked with the second series of the constructive triangles, concluding after many experiences that the figures are equivalent because they are composed of the same pieces.

NOW we find out WHY these figures are equivalent. The reason is that a certain relationship prevails in the lines.

The presentations of these insets of equivalence follow a certain pattern consisting of four periods which are valid for all of the studies:

1) Recognition of the two figures in the frame: not only the red ones where the figure is present, but also of that figure seen in the empty frame.
2) Proof of the equivalence between the two figures: this is done by interchanging the pieces.
3) Precise identification of the lines of each figure: base, altitude, etc.
4) Relationship between those lines.

THE INSETS OF EQUIVALENCE. . .
Introduction. . .

The first period---the recognition of the figures---is a sensorial experience, much like that which the child encountered in the work with the geometry cabinet. The second period---the proof of equivalence---is still a sensorial experience, recalling the work with the second series of constructive triangles. The third period---the precise identification of the lines of the figures---is a work already mastered by the child through the work with the geometry nomenclature. The fourth period---the relationship between the lines---is the new part. And it is new only in a certain aspect because it includes all the information gathered in the previous three parts.

So the fourth period of this study can be compared to the third part of the three-period lesson in which the child gives back what he has been given. Thus this fourth period is the fundamental study which takes the child directly to the study of area and the formation of formulas written with symbols. "This will lead to finding the area of different geometric forms and also to an intuition of some theorems." (Montessori, Elementary Materials.)

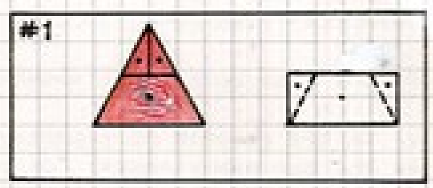
The work with these insets is presented in GROUP PRESENTATIONS, that is probably 4 or 5 children. There should be an interval of 4 or 5 days between the presentations, even though they involve a progression of corresponding ideas, in order to give the children time to experiment with their hands.

It is more important that the teacher see the needs of the child, to know when he is ready to work and when he needs more time for a certain work, than it is for her to know all of the material in its complexity.

Presentation #1: Showing that THE TRIANGLE is equal to a rectangle which has one side equal to the base of the triangle, the other side equal to half of the altitude of the triangle.

1. RECOGNITION: Identify the two figures. NOTE: We show an equilateral triangle here, unfortunately, as it is a particular case rather than the general. The triangle is divided first by joining the mid-points of the oblique sides, thus forming an isosceles trapezoid below this parallel line to the base and a similar triangle to the larger one above. Then one of the altitudes of this second triangle is traced, giving two right-angled scalene triangles.

1. This is a triangle. This is a rectangle.



2. PROOF OF EQUIVALENCE: we try to fit all the parts of the triangle into the rectangular space.

2. All of the parts of the triangle fit perfectly into the rectangle frame, so the triangle is equivalent to the rectangle. Why are they equivalent? Because I used the same number of pieces.

3. IDENTIFICATION OF LINES: (first discussing completely one figure and then the second)

   A. Replace the pieces in the triangle.

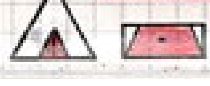
   Superimpose the two small right-angled triangles on the trapezoid: Remove one to show more clearly the altitude.

3A. In the triangle the line traced from the vertex to the mid-point of the base is the altitude. Now we divide the oblique sides at the mid-points, thus dividing them into two equal parts. AND we have thus also divided the altitude into two equal parts. Therefore, we see that the altitude of the smaller triangle is equal to the altitude of this trapezoid.

   B. Identify the base and the altitude of the rectangle.

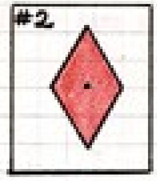
4. RELATIONSHIP OF THE LINES: Show the trapezoid base on the rectangle base. Now show the small triangle as half the altitude of the triangle again and then move the figure to the rectangle.

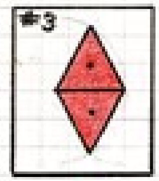
4. The base of the triangle is equal to the base of the rectangle. The altitude of the rectangle is equal to half the base of the altitude of the triangle.



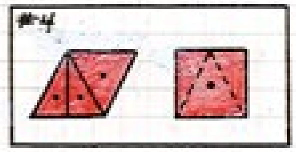
THE INSETS OF EQUIVALENT FIGURES. . .

Presentation #2: A RHOMBUS is equal to a rectangle which has one side equal to one side of the rhombus and the other equal to the height of the rhombus.





1. IDENTIFICATION: Rhombus, rhombus (for the second rhombus change the position of the parts to identify it as a rhombus) Rhombus. <u>Rectangle</u>.
NOTE: The rhombus present in the second frame is special: that one constructed with two equilateral triangles, as those found in the constructive triangles. The rhombus which is divided into three parts is formed by an equilateral triangle and two right-angled scalene triangles, one side of which forms the base of the rhombus.
NOTE: Here there are no empty frames. There are several ways to calculate the area of rhombi, depending on the position of the rhombus. THUS THE IMPORTANCE OF THE VERTICAL POSITION OF THE LONG DIAGONAL IN THE FIRST TWO AND THE HORIZONTAL POSITION OF THE SAME IN THE THIRD.



2. PROOF: Here the first and last figures, the whole rhombus and the whole rectangle, are the terms of comparison. The middle two are the mediators. Thus we begin by eliminating #2 and #3, arriving finally at the point where we can use only the two figures in #4 for the comparison.

NOTE: The arrangement as shown in #4 of the parts of the rhombus is important.

REMOVE THE RHOMBUS---the rectangle will not fit in the space. Show the pieces of the rhombus in the rectangle space.

2. We begin then with the first and second insets (#2 and #3), verifying congruency by exchanging the pieces.
NOW WE CAN REMOVE THE FIRST FIGURE (#2) and use the mediator, #3, the rhombus divided into two parts.
Secondly we look at the rhombus in #3 and that in #4: Do they have something in common?
Verify congruency.

The rhombus cut in a special way will fit into the rectangle space.

CONCLUSION: The rhombus is equivalent to the rectangle.

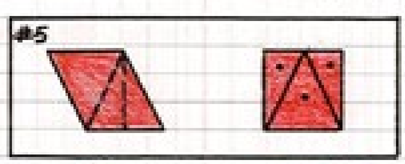
3. IDENTIFICATION OF THE LINES.

3. The base of the rhombus is one of its sides.
Show the base of the rectangle.

4. RELATIONSHIP BETWEEN THOSE LINES.
Show that the bases and altitudes are equal. NOTE: Here the position of the right-angled triangles in the rhombus is important as it allows us to identify the altitude of the rhombus.

4. The base of the rhombus is equal to the base of the rectangle.
The altitude is the rhombus is equal to the altitude of the rectangle.
THE RHOMBUS AND THE RECTANGLE ARE EQUI* VALENT BECAUSE THEY HAVE EQUAL BASES AND EQUAL ALTITUDES.

Presentation #3: The Equivalence of the Common Parallelogram and the Rectangle.



1. IDENTIFICATION: Common parallelogram and rectangle. (These two look like a rhombus and a square: it is important that their identification be parallelogram and rectangle.) SHOW THE PARALLELOGRAM BY ROTATING THE ANGLES---SIDES ARE NOT EQUAL.

2. PROOF: Take the parallelogram out. Fit the pieces of the rectangle in that frame. IMPORTANT TO SHOW PARTS IN THIS WAY:

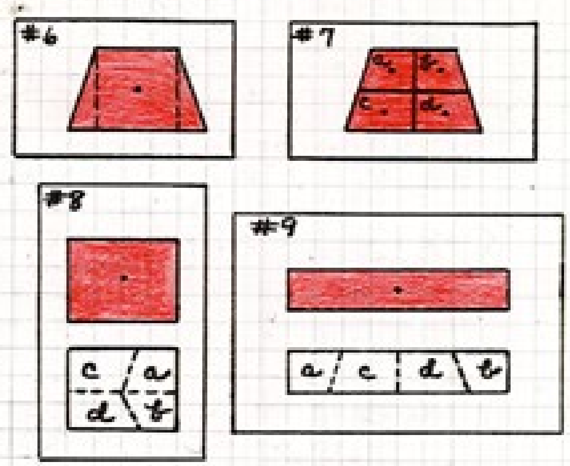
2. The common parallelogram is equi-valent to the rectangle.

THE INSETS OF EQUIVALENCE. . .
Presentation #3: Common Parallelogram and Rectangle. . .

3. IDENTIFICATION OF THE LINES: Using the large triangle, show that the two altitudes
4. RELATIONSHIP OF LINES: are equal and that the bases of the two are equal.

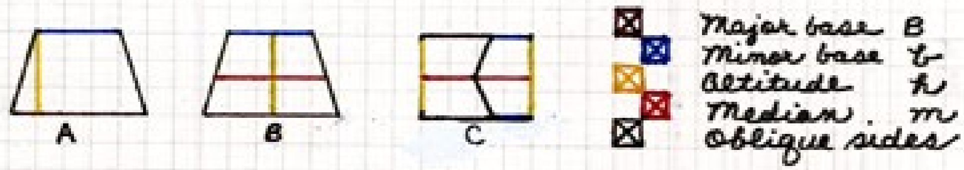
Presentation #4: The Equivalence of a trapezoid and a rectangle having one side equal to the sum of the two bases and the other equal to half the height.

1. IDENTIFICATION: The whole trapezoid. In the whole trapezoid we have a major base of 10 cm. The trapezoid subdivided by the altitudes gives a square in the middle whose sides are 6 cm. Thus we have a special isosceles trapezoid. Obtuse angles are 108°, acute angles are 72°.
The subdivided trapezoid. Here the trapezoid frame is equal. . .to the first. The mid-points of the oblique sides have been joined by a line that is perpendicular to the base. (This is sometimes called a median.) This median cuts the trapezoid into two isosceles trapezoids. Then the mid-points of the parallel sides are joined giving two right-angled trapezoids which are, in turn divided into four right-angled trapezoids. SHOW THAT THE TOP TWO ARE EQUAL AND THE BOTTOM TWO ARE EQUAL. (#7)



1. Trapezoid. . .trapezoid. . . rectangle. . .rectangle. . .

Guide figures for the Period 4 Relationship:



REFERENCE: Classified Nomenclature: F6

2. PROOF OF EQUIVALENCE: Here we have two proofs: 1) Trapezoid with Rectangle #8.
2) Trapezoid with Rectangle #9.
As a consequence, the two rectangles will be equivalent. #7 is the mediator.

1) Trapezoid with Rectangle #8
   A. Exchange the trapezoids of #6 and #7. They are congruent. The only difference between the two is the #7 has been divided into four parts.

   B. Remove #6: Show #7 and #8 for the first equivalence proof.

   C. With inset #8 exchange the rectangle into the empty frame showing that we have the same rectangle.

   D. Note that we cannot fit the rectangle into the trapezoid frame, so we must fit the pieces of the trapezoid into the rectangle frame.

   E. Ask the child to fit the pieces in without instructing him. There are, in fact, many ways to fit the pieces in, some acceptable and some not. It is important that the pieces are arranged here so that we do not show all of the fractional portions of the major and minor bases of the trapezoid internally. If the child does this, we help with the rearrangement.

   F. CONCLUSION: This trapezoid is equivalent to this rectangle.

2) Trapezoid with Rectangle #9
   A. Remove the trapezoid pieces from rectangle #8 and place again in the trapezoid figure: then show #7 and #9.

   B. Ask the child now, having shown both the rectangle in this frame to be the same, to fit the pieces of the trapezoid into this long rectangle. Arrangement here is important, too: Show as drawn in the inset figure above or the inverse position. (upside down) This is most successfully done by first taking the lower trapezoid pieces first and placing together center

C. CONCLUSION: This trapezoid is equivalent to this rectangle.

3. IDENTIFICATION OF THE LINES:

3. Trapezoid: minor base, major base, altitude.
Rectangles: base, altitude.

4. RELATIONSHIP BETWEEN THOSE LINES:

1) Between the lines of the trapezoid and rectangle #8.
2) Between the lines of the trapezoid and rectangle #9.
3) Between 2 rectangles.

1) The trapezoid and rectangle #8.

A. Have the child arrange the pieces of the trapezoid in the rectangle as shown in guide figure C. (Without this arrangement we find that we must give the corollary of the theorem first because the median of the trapezoid will be shown as the base of the redtangle.)

B. Note with the child that the base of the rectangle is formed by 2 elements of the lines of the trapezoid. SHOW ONE PIECE OF THE TRAPEZOID WHICH FORMS THE BASE OF THE RECTANGLE IN THE TRAPEZOID: it shows us that we have ½ of the major base corresponding to the base of the rectangle. SHOW THE SECOND PART OF THE RECTANGLE BASE IN THE TRAPEZOID: we have ½ of the minor base corresponding to the base of the rectangle.

C. CONCLUSION: The base of the rectangle is equal to ½ the minor base plus ½ the major base of the trapezoid.

$$\frac{(B+b)}{2} = \frac{B}{2} + \frac{b}{2} = B_R$$

NOTE: Now the child can see that the pieces must be arranged in a certain manner which will show this relationship between the lines of the figures. With the arrangement as in C we can show the relationship using both of the top parts which form the rectangle or both of the lower parts. Each gives $\frac{B+b}{2}$

D. Using now the two pieces which form the rectangle's base, reverse to show in the trapezoid frame that the sum of the two parts of the rectangle base (½ minor base and ½ major base) give a line which is equal to the median. DEFINE median: The median of the trapezoid is that line uniting the midpoints of the non-parallel sides.

E. CONCLUSION: THEOREM: The median of the trapezoid is parallel to the bases and equal to ½ their sum.

F. Compare the altitudes of the two figures: they are equal.

G. CONCLUSION: These two figures are equivalent because their altitudes are equal and the base of the rectangle is equal to ½ the sum of the bases of the trapezoid.

2) The trapezoid and rectangle #9

A. Show the parts of the trapezoid in the rectangle frame by taking first the two parts of the major base of the trapezoid, showing them in the center and then fitting the two upper parts of the trapezoid in the sides.

$$B_R = \frac{b}{2} + B_T + \frac{b}{2} = B_T + b$$

B. $\frac{b}{2} + B + \frac{b}{2}$

B. OR take the top two parts which form the upper trapezoid (minor base) and show them first in the rectangle frame. Then fit the other two.

$$B_R = \frac{B_T}{2} + b + \frac{B_T}{2} = b + B_T$$

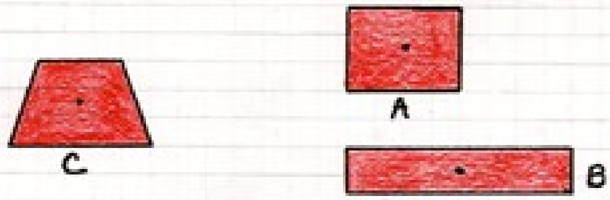
C. CONCLUSION: The base of this rectangle is equal to the sum of the bases of the trapezoid.

D. Show that the altitude of this rectangle is equal to ½ the altitude of the trapezoid by replacing the pieces in the trapezoid, showing each piece of one-half the trapezoid in the rectangle frame and then showing those two pieces to be each one-half of the altitude of the trapezoid.

E. CONCLUSION: These two figures are equivalent because the base of the rectangle is equal to the sum of the bases of the trapezoid and its altitude is equal to ½ that of the trapezoid.

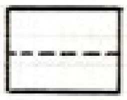
3) The two rectangles.

A. Show the whole figures from #6, #8 and #9: We have demonstrated the equivalence of C and A and why the two are equivalent; we have shown that C and B are equivalent and why. Since we have said that A = C
and B = C
then A = B.



B. We can prove this by weighing the two pieces. OR

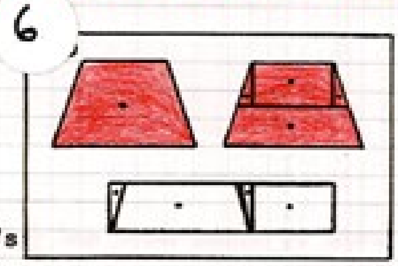
C. We can cut paper figures: a) Beginning with A, fold so that the longer opposite sides come together, then cut on the line. Form the longer rectangle and superimpose the metal inset on it to show that they are equal. b) Begin with B, folding it lengthwise, cutting and then rearranging the parts to form A. Superimpose the metal inset to prove equality.



D. Analyze the relationship of lines:
The altitude of A is equal to twice the altitude of B and its base is one-half the base of B.
The altitude of B is equal to one-half the altitude of A and its base is equal to twice the base of A.

6

Presentation #4: Part II: With inset #10



1. IDENTIFICATION: The trapezoid, the trapezoid divided into parts, the rectangle.

2. PROOF OF EQUIVALENCE: We note that the whole trapezoid cannot be fitted into the rectangle's frame, so we must use the same trapezoid divided for the proof:
   a) Interchange the trapezoids to verify congruency. — a) The trapezoids are equal, so we can use the second one divided for our proof of equivalence.
   b) Fill the empty frame with the four pieces of the trapezoid as shown in figure. — b) This trapezoid is equivalent to this rectangle.

3. IDENTIFICATION OF LINES:
   a) Trapezoid: We can identify the major base, the minor base, and the altitude which is a line drawn perpendicular to the base (we can imagine it as it is not shown).
   b) Trapezoid divided: The altitude of the whole trapezoid is cut in half by a line drawn parallel to the base. The result is two trapezoids. The upper trapezoid is divided by the two altitudes drawn from each vertex to the base, giving two little triangles and one rectangle.
   c) Superimpose the little triangle or the rectangle on the lower trapezoid — Each has a major and minor base. The line dividing the upper trapezoid is equal to $\frac{1}{2}$ the altitude of the whole trapezoid.
   d) Rectangle: Base, altitude

4. RELATIONSHIP BETWEEN LINES:
   a) Fit only the two large pieces of the trapezoid into the empty frame:
   The base of the rectangle is formed by the major base plus the minor base:
   $$R_b = B_T + b$$
   b) Now put the two small triangles in the remaining spaces, showing first that the small triangle is equal to $\frac{1}{2}$ the trapezoid. SO the altitude of the rectangle is equal to $\frac{1}{2}$ the altitude of the trapezoid. $a_R = \frac{1}{2} a_T$

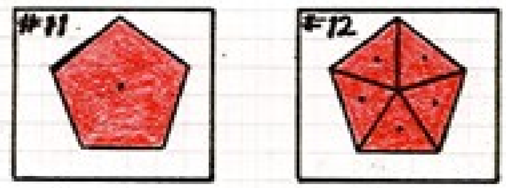
c) CONCLUSION: The trapezoid is equivalent to the rectangle because the base of the rectangle is equal to the sum of the bases of the trapezoid and its altitude is equal to ½ the altitude of the trapezoid.

NOTE: The difference in the work with inset #10 is that here the trapezoid is divided so that both the major and minor bases are left complete. Thus we are able to see their sum more easily. (p. 280, Montessori Elementary Materials)

Presentation #5: The Equivalence between a regular polygon and a rectangle having one side equal to the perimeter and the other equal to half of the hypotenuse or the <u>apothem</u>.

Material: Insets #11 - #16

Part I: The Regular Pentagon, insets #11, #12



1. IDENTIFICATION: Show that we have a regular pentagon by rotating it in the frame.

2. NOMENCLATURE OF THE PENTAGON: The knob indicates the <u>center</u> of the figure. Sides. Perimeter.
Remove the parts from the second pentagon: Classify the triangles. Verify congruency.

Imagine that we have joined the center to each of the five vertices.
Cutting along those lines we obtain 5 triangles found in #12. . .acute-angled isosceles triangles.
They are all congruent.

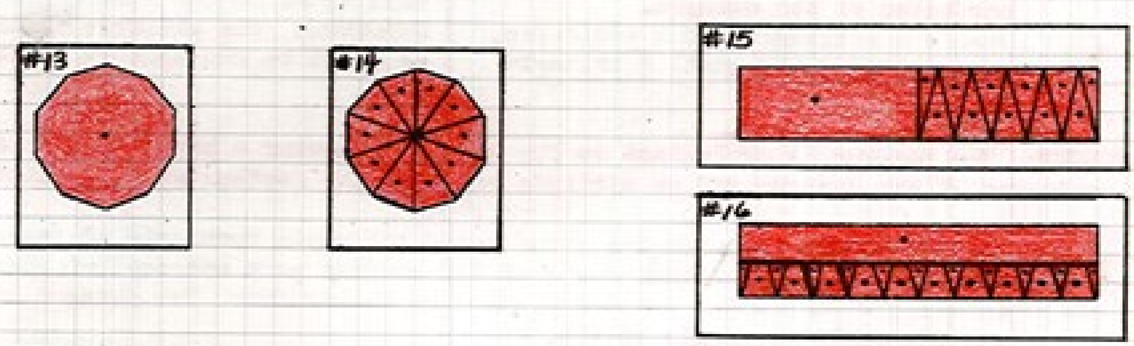
3. Interchange the pentagons, vertifying congruency between them. NOW WE CAN REMOVE THE PENTAGON #11; we have achieved the first mediator (#12).
NOTE: This is information for further reference.

4. STATEMENT: Any regular polygon can be divided into as many equal triangles as it has sides.

Part II: Division of the Decagon

While the pentagon is not developed in all its consequences, the decagon is now thoroughly developed.

Material:



1. IDENTIFICATION: Regular decagon. The same decagon. A strange rectangle. And another strange rectangle. But the first is different from the second in the way it is divided. (Life two whole rectangles to emphasize the difference in the division.

2. PROOF OF EQUIVALENCE: Between decagon and whole rectangle in inset #15. ~~whole triangle in inset #15~~.
   a) Show the two decagon insets #13 and #14. Explain the division: we have obtained 10 acute-angled triangles by joining the center with each of the vertices of the regular polygon with ten sides, the decagon. Show = of triangles.
   b) Interchange the decagons to prove congruency. Eliminate #13. Now #14 is the mediator.

THE INSETS OF EQUIVALENCE. . .
Presentation #5: Part II: Division of the Decagon. . .

c) Take inset #15. Superimpose the whole rectangle on the second half formed of pieces. State congruency. Set the whole rectangle aside.

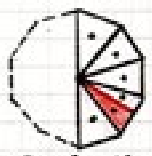
d) From #14 transfer the parts of the decagon to the space left by the whole rectangle in #15. BEGIN BY SHOWING FIVE OF THE TRIANGLES (½ the decagon). AT THIS POINT WE HAVE FORMED THE ½ BASE OF THE RECTANGLE INSET. Then rearrange the half-triangle piece in the rectangle to fill the space lacking at the corner and finish transferring all triangles of the decagon into the rectangle.

e) CONCLUSION: There are in this long rectangle the equivalent parts of two decagons. SO. . .this rectangle (whole which has been set aside) is equivalent to the decagon.

SHOW THE DECAGON FROM #13 and THE WHOLE TRIANGLE FROM #15.

f) Prove the above statement by replacing all of the small pieces now shown as the rectangle in the two decagon frames.
CONCLUSION: One decagon is equivalent to ½ this rectangle.
This rectangle is equivalent to two decagons.

3. IDENTIFICATION OF LINES: The decagon:
NOTE: The reason for the ½ triangle in the decagon is that it gives us the line which joins the center of the decagon with the midpoint of one side: thus the apothem.



Show the whole rectangle in the frame:

1. Center. Side.
The decagon has no altitude.
But, if we look at this one triangle in the decagon which has been divided in half, we find the altitude of the triangle---thus the altitude of each of the triangles.
This line, in a regular polygon, which joins the center to the mid-point of one of the sides is called the apothem.
Base. Altitude.

4. RELATIONSHIP OF LINES:

a) Replace the decagon pieces into the empty half of the rectangle frame now, the other half occupied by the whole rectangle. BEGIN WITH ONLY FIVE TRIANGLES, THUS FORMING THE BASE OF THE RECTANGLE (½ the whole rectangle frame) WITH ONE-HALF THE PERIPHERY OF THE DECAGON' (5 SIDES)

b)

b) GIVE THE CONCEPT OF PERIMETER:

CONCLUDE: The base of the rectangle is equal to one-half the perimeter of the decagon.
Show the altitude of the rectangle, superimposing the small ½ triangle on it.

b) The perimeter is the total length of the sides of a polygon.
How many sides form this perimeter?
How many sides have we used to form the base equal to the base of this rectangle
CONCLUDE: the altitude of the rectangle is equal to the apothem of the decagon.

STATEMENT: The decagon is equivalent to the rectangle having a base equal to one-half the perimeter of the decagon and an altitude equal to the apothem of the decagon.

****

7A. PROOF OF EQUIVALENCE: Between the decagon and the whole rectangle of inset #16

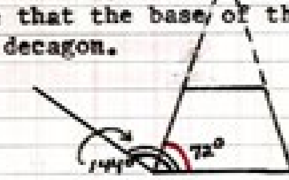
a) Superimpose the whole rectangle of #16 on that part composed of pieces to verify congruency. NOTE THE TERMS OF COMPARISON: The decagon of #13 and the whole rectangle of #16.

b) Transfer first the small trapezoids of the rectangle to the frame of the decagon (that decagon whose ten triangles are still shown as ½ of the rectangle frame of the prior presentation). We obtain two decagons. Then complete the passage with the final 11 pieces of the rectangle moved to the interior decagon, finishing with the two tiny halves of the triangle.

c) The decagon is equivalent to the whole rectangle.

d) Prove the statement by replacing the pieces in the rectangle. NOTE: with only the trapezoids in the rectangle again, we see that the base of the rectangle is formed by the whole perimeter of the decagon.

NOTE: The major base of the trapezoid as seen in insets #6, #7, and #10 is equal to the side of the decagon. Base angle of trapezoid is 72o
Angles of decagon equal to 144o.



3A. IDENTIFICATION OF LINES: The decagon: Perimeter, sides, center, apothem.
Rectangle: Base, altitude.

4A. RELATIONSHIP BETWEEN LINES: Using the tiny half of one of the triangles from this set of pieces, show that the altitude of that triangle which it cuts in half is equal to the apothem of the decagon. (by using both one can show this superimposed on one of the triangles of the decagon. . .by taking the ½ of the decagon triangle used in the prior proof, the two tiny triangles can be shown clearly to be half of the apothem.) THEN SHOW THAT THIS ONE-HALF THE APOTHEM (the tiny triangle) IS EQUAL TO THE ALTITUDE OF THE WHOLE RECTANGLE.

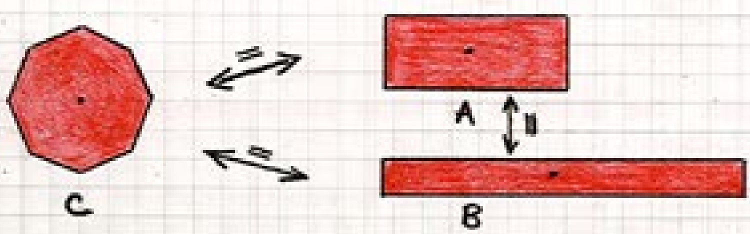
The altitude of each of the triangles which compose the decagon has been divided in half by a line parallel to the base. SO the trapezoid and the tiny triangle have equal altitudes. (SHOW THE DECAGON TRIANGLE AND THE TWO PARTS WHICH FORM IT.) THEN. . .each of the pieces which compose the half of the rectangle frame (that is, which form a rectangle equal to the whole) is equal to ½ the apothem of the decagon BECAUSE THE APOTHEM IS EQUAL TO THE ALTITUDE OF THE TRIANGLE WHICH FORMS THE DECAGON.

CONCLUSION: The altitude of the rectangle is equal to ½ the apothem of the regular polygon.

By replacing only the trapezoids in the rectangle we formed the base. So we see that the base of the rectangle is equal to the perimeter of the regular polygon.

THIS DECAGON is equivalent to the RECTANGLE having as its base the perimeter of the polygon and as its altitude one-half the apothem of the decagon.

5. SHOW THE THREE WHOLE FIGURES: We have proved that this rectangle is equivalent to the decagon; and that this rectangle is equivalent to the decagon. So the TWO RECTANGLES ARE EQUIVALENT TO EACH OTHER.



6. REMOVE THE DECAGON: Review the relationships of each base and altitude of the rectangles to the decagon. CONCLUDE: The base of rectangle A is equal to ½ the base of rectangle B and its altitude is two times that of B. OR The base of rectangle B is equal to two times the base of A and its altitude is ½ that of rectangle A. THE TWO RECTANGLES ARE EQUIVALENT.

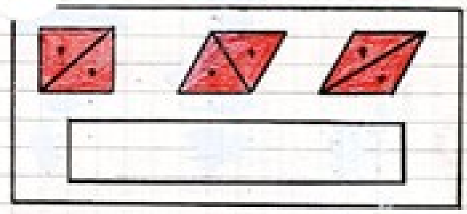
7. Prove the statement by using a paper-cutting exercise. The child begins with one of the rectangles, folds A so that the longer bases meet, cuts on the line, then repositions the two pieces to show B. Superimposes the inset on the new figure to verify congruency.

THE INSETS OF EQUIVALENCE. . .

Presentation #6: All triangles having the same base and altitude are equivalent.

NOTE: Here we do not use the rectangle as the second term of comparison as we have in the previous equivalence proofs. The rectangle frame shown in inset #17 is only a reference point in which we show the equal alti[illegible] of the three triangles.

11

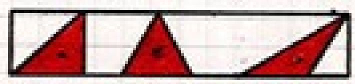


Material: Inset #17. Here we have three figures: figures plus the empty rectangle frame. The first rhombus is divided into two equilateral triangles, the rhombus being divided along the minor diagonal. The second rhombus has been divided along the long diagonal. The rectangle is divided by one diagonal which gives two right-angled triangles.

Presentation

1. IDENTIFICATION: Rectangle. Rhombus. Rhombus.

2. PROOF OF EQUIVALENCE:
   a) Interchange the rhombi to verify congruency. We see that the only difference is the way they have been divided. Show equivalence of first two figures.
   b) Take one triangle from each figure and classify: Right-angled triangle, Acute-angled triangle, Obtuse-angled triangle.
   We see that we have the three types of triangles as classified by angles that we discovered in the geometry cabinet. <u>Show</u> the three insets as a comparison.
   c) Superimpose each of the three triangles on its corresponding half, verifying congruency between each of the pairs.
   d) Give each triangle a fractional value as related to its whole figure: $\frac{1}{2}$.
   e) Show in the empty frame $\frac{1}{2}$ of each figure, that is, one of each of the three triangles.



   f) CONCLUSION: The altitude of these three triangles is the same. (IN ORDER TO MAKE THIS STATEMENT, IT IS IMPORTANT TO SHOW THE OBTUSE-ANGLED TRIANGLE AS IN THE FIGURE ABOVE.)
   g) Verify the bases as being equal by showing each of the three triangles in each of the three empty frame figures on the top row. In each case, we know that it fits in one base and then we show that it also fits in the other two.
   h) CONCLUSION: All the triangles have the same base.

STATEMENT OF EQUIVALENCE: Since we know that the three figures at the top (the two rhombi and the rectangle) are all equivalent to each other, we can say that one-half of each of those figures is equivalent to the other halves.

AND We see that the three halves, these three triangles, have the same base and altitude---the three have equal bases and altitudes.

THEREFORE we can say that triangles which have equal bases and equal altitudes are equivalent to each other.

AND because each of one-half of an equivalent figure, the halves are equivalent to the other halves. That is, the two right-angled triangles are equivlant to the two acute-angled triangles. The two acute-angled triangles are equivalent to the obtuse-angled triangles. The two obtuse-angled triangles are equivalent to the two right-angled triangles.

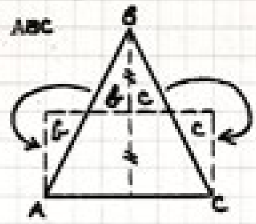
RESEARCHES AND EXERCISES TO FOLLOW OR SUPPLEMENT THE INSET EQUIVALENCE WORK:

A. The child repeats individually the presentations given by the teacher.

B. The teacher prepares various figures: triangles, rhombi, parallelograms, trapezoids, polygons with more than four sides. THEN. . .

B. . .ask the child to prepare the rectangle equivalent to each figure. Example:

From triangle ABC



He constructs the equivalent rectangle, cutting the triangle as shown as a result of his experience with the insets. He resets the pieces to form the equivalent rectangle. . .AND HE MAY TRANSFORM IT THEN INTO THE LONGER RECTANGLE.

C. The teacher indicates the measurements of a figure and asks the child to construct the rectangle equivalent to that figure. The measurements can be indicated:
1) in units, 2) in units of the metric system, 3) in units of English system.

Example: The triangle I want you to construct has a base of 4 units and an altitude of 8 units. In order to construct the equivalent triangle, one first must DECIDE WHAT THE UNIT LENGTH WILL BE. He is now working in area without any system of measurement.

NOTE: We see here the necessity for the child to carefully record each of the equivalence statements he has reached in his work with the insets of equivalence. As he comes to this activity, he may refer to those proofs as a guide for his construction on the basis of those statements.

THE PYTHAGOREAN THEOREM

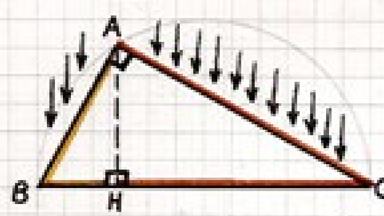
The materials used for the presentation of the Pythagorean theorem here are insets #18, #19 and #20, as shown in the following presentation figures (as they appear in the actual presentation.) Inset #18 presents the sensorial demonstration of the theorem, #19 is used for the arithmetical demonstration of the theorem, and #20 presents the Euclidean theorem, the geometrical demonstration. The work then proceeds: 1) the presentation with the isosceles triangle, 2) the presentation with the Pythagorean triple, 3) the presentation with the Euclidean theorem, and 4) the extensions of the Pythagorean theorem, already examined in the second constructive triangle work.

INTRODUCTION (for the teacher)

As the Pythagorean theorem is based on the theory of equivalence, it is the conclusion of this work. The three frames could be examined in isolation, but the comprehension would be lessened. In this presentation of the frames in conjunction with each other, we can refer to experiences in our presentation that have already been examined. Particularly sophisticated is frame #20. It is based on the Euclidean theorem, to precisely demonstrate the first theorem of Euclid which is:

In each right-angled triangle, one leg is the mean proportion between the hypotenuse and the projection of that leg on the hypotenuse.

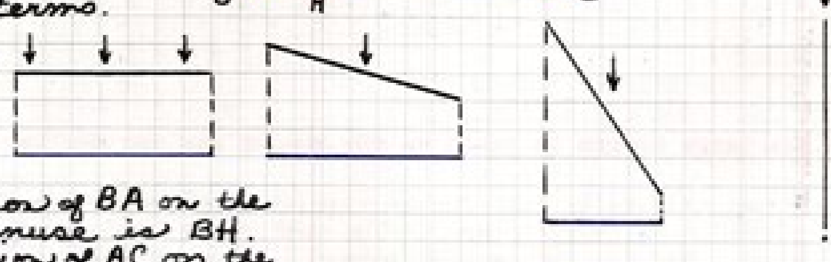
BC : AB :: AB : BH



and BC : AC :: AC : HC

There are four terms in the proportional: two middle terms (the mean) and two external terms.

Projection:



(arrows indicate direction from which light source originates)

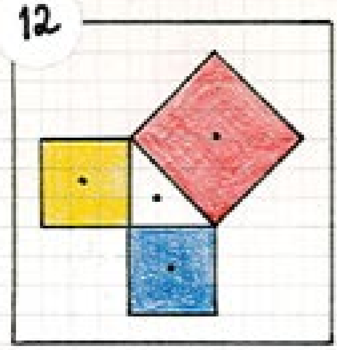
. . . projection of BA on the hypotenuse is BH.
. . . projection of AC on the hypotenuse is HC

And . . . In any right-angled triangle, the height is the mean proportion between the projection of both legs on the hypotenuse

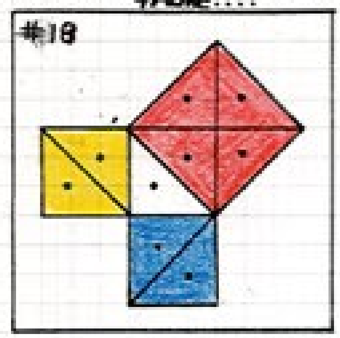
so BH : AH :: AH : HC

THE INSETS OF EQUIVALENCE. . .
Pythagorean Theorem. . .

Presentation #1: The Pythagorean Theorem: Insets #18, #19



....two together form one frame....



1. Using inset #18, isolate the right-angled triangle: the only triangle. IDENTIFY: it is a right-angled triangle. It must be a right-angled triangle because this is our fundamental element. It is an isosceles triangle. (This makes the demonstration of the theorem much easier because the triangle is also half of the two smaller squares.)

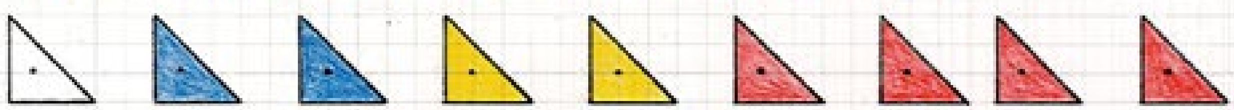
2. VERIFY THE EQUALITY OF THE YELLOW AND BLUE SQUARES.

3. Using inset #18, show the same triangle in the center. NOTE THAT IT IS CONGRUENT TO THE FIRST TRIANGLE OF #18.

4. Show the congruency between the two blue triangles which form one square. Then that the two yellow triangles are congruent. THEN SHOW THAT ALL FOUR ARE EQUAL TO EACH OTHER.

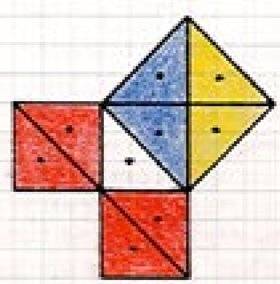
5. Show the division of the red square. SHOW THAT ALL FOUR TRIANGLES ARE CONGRUENT TO EACH OTHER: AND THAT THEY ARE CONGRUENT TO THE YELLOW AND BLUE TRIANGLES.

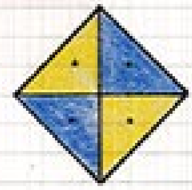
6. Show that we have nine equal right-angled isosceles triangles:



7. CONCLUSION: We have proved that the two figures in the frame #18 are equal; and that all the triangles which form the figure in the second one shown are equal among themselves.

8. Now we want to show THAT THE SUM OF THE TWO SMALL SQUARES IS EQUAL TO THE BIG SQUARE: Frame #2 is the mediator.
   a) Remove the red triangles from the large square. Fit the yellow and blue triangles from the small squares into the big square.
   b) Rearrange for interesting contrast.
   c) CONCLUSION: The two small squares together are equal to the big square.
   d) Replace the four red triangles in the two small squares. NOT NECESSARY FOR THE PROOF: BUT A NICE COMPLETION OF THE EXERCISE.





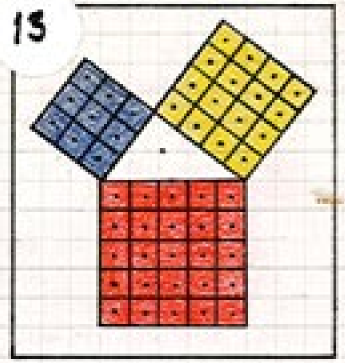
9. MATHEMATICAL PROOF: The child counts the ~~squares~~ triangles in the large square; then he counts the two small squares.
NOTE: We can take the triangles as a unit because we know that they are all equal.

9. $1 + 1 + 1 + 1 = 4$ AND
$(1 + 1) + (1 + 1) = 4$ SO
$1 + 1 + 1 + 1 = (1 + 1) + (1 + 1)$
$4 = 2 + 2$ OR
$2 + 2 = 4$

THEN the large square is equal to the sum of the two smaller ones.

NOTE: This material was originally constructed with the smaller squares having the same color. THEN if all square were of the same color, one could more easily show equal areas. But the three-color scheme is now utilized because it is a preparation for the next presentations.

THE INSETS OF EQUIVALENCE. . .
THE PYTHAGOREAN THEOREM. . .



Presentation #2: Frame #19: Numerical proof

Material: Inset #19 shows the right-angled triangle with the squares built on each side, as in inset #18. The colors used are the same; but the positioning in the frame is different. THE BIG DIFFERENCE IS THAT HERE WE HAVE THE RIGHT-ANGLED SCALENE TRIANGLE WITH SIDES 3, 4, and 5.

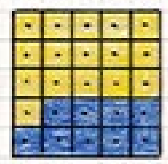
1. Begin by displaying both insets #18 and #19. Position #18 on a diagonal so that the two figures are in the same position as the figure shown in #19. With the identical positions, we note the same figure in both frames.

2. Take the two right-angled triangles out of the center of #18 (one figure) and #19; superimpose the two to show that THEY ARE NOT CONGRUENT.

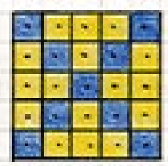
3. Ask the child to classify this new triangle: It is right-angled (like the triangles in #18); but it is SCALENE, not isosceles.

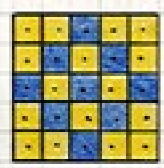
AND

4. By counting the sides of the squares that are lying on each side of the triangle: minor leg, major leg, and hypotenuse, we find that the minor leg is divided into 3 parts, the major leg is divided into 4 parts and the hypotenuse is divided into 5 parts. THE SIDES ARE IN A RELATIONSHIP OF 3, 4, 5. This is the 3, 4, 5 relationship which is the fundamental Pythagorean theorem: the classical triple of geometry.

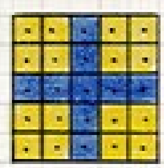


5. Count the squares: it is necessary to count only one side and then to square that number to get the total.
   a) Count the red square first and remove them from the frame: 5 X 5 = 25 squares.
   b) Place the yellow squares in the red square, counting them: 4 X 4 = 16.
   c) Count the blue squares: 3 x 3 = 9, and fill up the rest of the large square frame.
   d) THE LARGE FRAME IS FILLED: the proof is made: the sum of the two smaller squares is equal to the larger square.
   e) Replace the red squares in the two smaller squares to complete the exercise.





6. Arrange the new bi-color square symmetrically: the child can copy the patterns in his notebook. INVITE THE CHILD TO FIND AS MANY NEW WAYS TO ARRANGE THE TWO COLORS OF SQUARES AS POSSIBLE.



7. Arithmetical proof:

$$3 \times 3 = 3^2 = 9$$
$$4 \times 4 = 4^2 = +16$$
$$5 \times 5 = 5^2 = 25$$

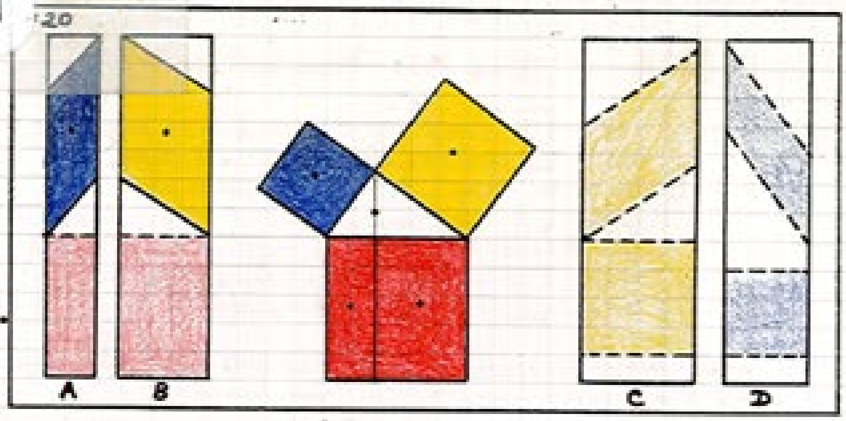
NOTE: The child should know that $3 \times 3 = 3^2$. This particular notation is introduced in the powers and is particularly important here.

EXERCISE: Invite the child to do this arithmetical proof with the list of triples, as shown in the introductory comments to the box of sticks.

NOTE: For this presentation, the inset should be prepared before the presentation. However, due to the large number of very small pieces, a special box in which these small pieces are kept may be advisable.

Presentation #3:

Frame #20: Euclidean Theorem



Frame #20 is composed of three parts:

1) The middle part is occupied by the right-angled scalene triangle (3,4,5,) and the corresponding squares on each side. The construction of this figure is the same as that shown in frame #19

2) The part on the left side is formed of two frames, rectangles A and B. The length of these two rectangles is not particularly significant (although, as we see, the parallelogram and the small rectangle fit exactly lengthwise in the first: that is, the length is equal to the long base of the parallelogram plus the long base of the small rectangle). The width of the rectangles is important, each width given by the altitude of the corresponding parallelogram which is shown in the rectangles in the figure---and also in the frame itself, in this position, on first presentation.

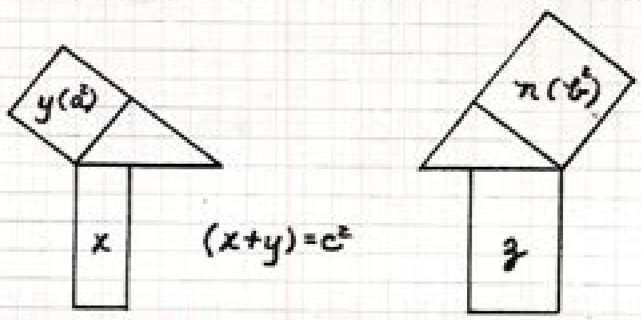
3) The part on the right side is also formed of two rectangles. And again it is the width which is of importance. The width of rectangle C is equal to the side of the large yellow square and also to the altitude of the yellow parallelogram with the short side as the base. The second rectangle D has a width which is equal to the side of the small blue square and also equal to the altitude of the blue parallelogram with the short side as the base.

The two parallelograms, blue and yellow, are the mediators in our proof of equivalence between the two rectangles (the result of the division of the square c) and the two squares, blue and yellow.

On the white right-angled scalene triangle the only internal altitude is traced as a black line which results in a division of the hypotenuse into two line segments: one line segment representing the projection of the minor leg on the hypotenuse and the second line segment representing the projection of the major leg on the hypotenuse.

The square on the hypotenuse (c) is divided into two rectangles by the prolongation of this interior altitude which falls on the hypotenuse.

NOTE: Large cardboard pieces which correspond to each part of this frame are helpful in showing this proof. Needed are the two parallelograms, the two squares, the two rectangles, the white triangle divided into two parts by the altitude, the four long rectangles which represent the empty frames, and finally, two large cardboard backup pieces which are joined to make the figure in the center.



We must prove that the rectangle x is equivalent to the square y; and that the rectangle z is equivalent to the square n. THEN y + n will be equivalent to x + z OR

$$a^2 + b^2 = c^2$$

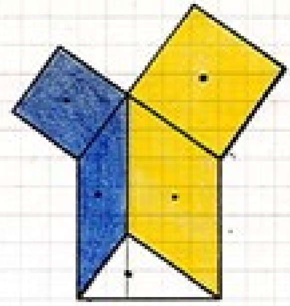
1. To prove that the squares are equivalent to the rectangles, we can try to fit the rectangle frames or vice versa; but it is impossible. So we must use the mediators.

2. Remove both rectangles, slide the triangle to the bottom of the large empty square frame; NOW WE CAN FIND A CERTAIN RELATIONSHIP WITH THE NEW SHAPES.

THE PYTHAGOREAN THEOREM. . .
Presentation #3; The Euclidean Theorem. . .

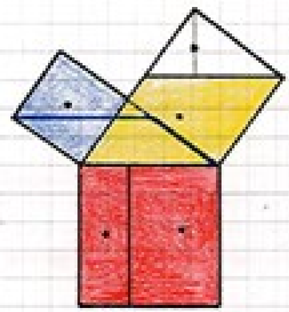
WE HAVE DEMONSTRATED WITH THE TWO MEDIATORS that the rectangles are equivalent to the parallelograms because the mediators fit exactly into the place of the rectangles.

EMPHASIZE their equivalence by holding a rectangle in one hand, a parallelogram in the other: they would weigh the same.



3. In order to go from the parallelograms to the squares, we remove the squares, one at a time, slide the triangle to the upper vertex of that square and fit the corresponding parallelogram into the vacated space.

WE HAVE DEMONSTRATED WITH THE TWO MEDIATORS that the squares are equivalent to the parallelograms because the mediators fit exactly into the place of the squares.



4. Replace all pieces in their original positions.

5. Note that the mediator is the common term:

rectangle = parallelogram
parallelogram = square

It is now possible to show the equivalence between the two squares and the two rectangles because they are both holding hands with the parallelogram. WE HAVE DEMONSTRATED THEIR EQUIVALENCE WITH THE TRANSIENT PROPERTY.

NOTE: Sensorially we have shown these equivalences by the exchange of the figures: it has been a visual, tactile experience. NOW we want to know the reason why the figures are equivalent. This is the same thing as looking for the RELATIONSHIP WHICH EXISTS BETWEEN THEIR LINES. The relationship of the lines is the last part of the sequence for each series of the inset equivalence work. And so it is here the last part of the proof of the theorem.

Presentation #3a: The Relationship of the Lines of Equivalent Quadrilaterals

1. The first consideration is the equivalence between the small rectangle and the ~~blue square (a)~~ blue parallelogram. Identify the lines of both figures: NOMENCLATURE. We note that each figure has a base and altitude but that the two elements can be switched in both figures: 2 bases and 2 altitudes, depending on the position of the figure. With the parallelogram note both the internal and external altitudes for both positions.

2. SHOW that the two figures must be oriented in a certain way for the proof.

3. Superimpose the two pieces to show that the altitudes are equal, and place the bases side by side to show that they are equal.

1. The rectangle has a base and altitude. But these two elements of the nomenclature can be switched if we show the rectangle in a different position. AND in the parallelogram there are 2 bases and 2 altitudes.

2. The two figures must be oriented in a certain way: We will use the long side of the rectangle as the base, and the long side of the parallelogram as the base.

3. The base of the rectangle is equal to the base of the parallelogram. The altitude of the rectangle is equal to the altitude of the parallelogram.

4. Fit both figures into frame A showing that both figures have the same altitude.

5. CONCLUSION: Because the rectangle and the parallelogram have equal bases and equal altitudes, they are equivalent.

6. The relationship of the lines between the larger rectangle and the yellow parallelogram is now considered. Give the nomenclature for the two figures, naming both bases and altitudes of each figure. Denote the base and altitude which will be considered: for both figures we consider the longer side as the base. Placing the two bases side by side, we show the the bases are equal. Superimposing the figures, we show that the two altitudes are equal. SHOW THE TWO FIGURES IN FRAME B TO VERIFY THE EQUALITY OF THE ALTITUDES. Because this rectangle and this parallelogram have equal bases and equal altitudes, they are equivalent.

NOTE: #4 and #6 placement of the figures in the long rectangles of the inset is shown in the figure of that inset; also the positions of the parallelogram in relationship to the squares shown in the rectangles C and D.

8. The blue square and the blue parallelogram: We again identify the nomenclatures first; stating that now we must reverse the orientation of the parallelogram for this consideration. The short side is now the base of the parallelogram. Show that the two figures have equal bases and altitudes by superimposition. Then show the two in rectangle D, verifying the equality of the altitudes. Because the square and the parallelogram have equal bases and equal altitudes, they are equivalent.

7. NOTE: We do this proof before the blues: an interesting observation on the arrangement of the rectangles: in the first set of two, we show the blue parallelogram comparison first, then yellow; in the second group, we have the yellow first, then the blue. The same figures then are placed in different orientation.

The yellow square and the yellow parallelogram: Identify the nomenclature; show that the short side of the parallelogram must now be considered as the base. Show the bases as equal and the altitudes as equal with the two figures superimposed. Then show the two in rectangle C, verifying the altitudes as equal. Because the square and the parallelogram have equal bases and equal altitudes, they are equivalent.

9. Review the relationship comparisons which have been made. Using the transitive property, restate the result of the equivalences.

rectangle = parallelogram
‖
square =

9. The small rectangle is equivalent to the small parallelogram.
The large rectangle is equivalent to the large parallelogram.
The large square is equivalent to the large parallelogram.
The small square is equivalent to the small parallelogram.

10. PASSAGE FROM THE RECTANGLE TO THE CORRESPONDING SQUARE: (the square and the rectangle are not commeasureable: there is no common measure.)

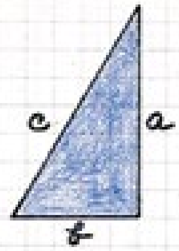
AGES: Pythagorean theorem: #18 inset: 8½
The study of area and the metric system must be considered before. . .
Arithmetical proof: #19 inset: 8½
Then the child is ready to work in this interim with the triples and extensions.
The Euclidean proof: #20 inset: 11

The Pythagorean Theorem... **Age : 12**

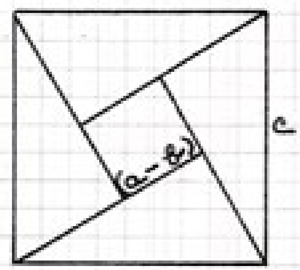
Presentation #4: **Algebraic Proof**

Material

1. The 12 equal right-angled scalene triangles from the Constructive triangles, series #1, box #3.
2. The operation signs
3. Paper slips



1. Ask the child to classify the triangle: **it is a right-angled scalene triangle.**
2. Identify the major leg, minor leg, and hypotenuse: Label a, b, and c.
3. Take 3 additional triangles: superimpose the four to verify congruency and note that the corresponding sides can all be designated as a, b, and c. **All these hypotenuses are c...**
4. Ask the child to construct the 4-pointed star; then pass to the diaphragm.
5. Apply to each of the triangles the nomenclature a, b, and c. Point out the position of each of these lines.



6. Identify the line which forms the side of the internal square: **(a − b)**

   Show that on the mat with symbols and letters and parentheses.
7. Point out that the side of the big square is c. Therefore, the area of the large square is c^2.

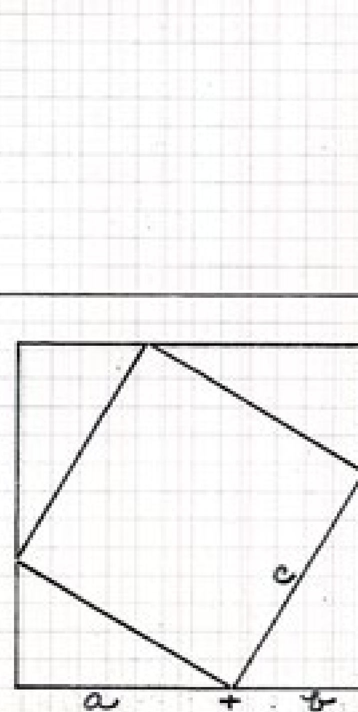
   and c^2 = the area of the 4 triangles + the area of the internal square.
8. Identify area of the triangles: one triangle is $\frac{1}{2}bh$ OR... $\frac{ab}{2}$

   THEN the area of the four triangles is **$4\frac{ab}{2}$**
9. Show the calculation on the mat:

$$c^2 = (a - b)^2 + 4\frac{ab}{2}$$

$$c^2 = a^2 - 2ab + b^2 + 2ab$$

$$c^2 = a^2 + b^2$$



Second proof: Form following square with the triangles. Identify a, b, and c of the triangles. Identify the side of the external square and the side of internal. Show the calculation on the mat:

$$(a+b)^2 = c^2 + 4\frac{ab}{2}$$

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

CANCEL

by placing the $-2ab$ on top of $2ab$ (− and +)

THEN $a^2 + b^2 = c^2$