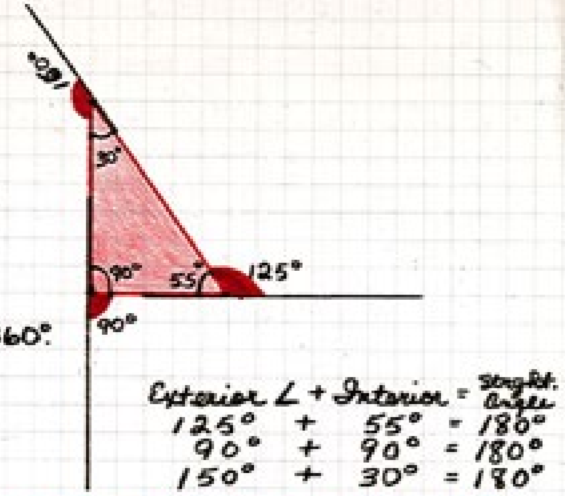
THE SUM OF THE EXTERIOR ANGLES OF THE POLYGON. . .



EXERCISE: Using all the polygons from the geometry cabinet and the transparent protractor.

a) With the protractor, the child measures the interior and the exterior angles. (He measures the interior angles as a CONTROL because he knows that the exterior angle and its corresponding interior angle are always equal to a straight angle. So he knows the total must be 180°)

b) He adds the exterior angles: 125°+90°+150° = 360°.

c) At the end of all the experiences he reaches the CONCLUSION:

The sum of the exterior angles of a polygon formed by extending each side in succession is two straight angles---or a whole angle.

AGE: The first year of Secondary School.

THE GOLDEN CHAPTER OF GEOMETRY FOR THE ELEMENTARY SCHOOL: LEVEL III: Congruence, Similarity, Equivalence

INTRODUCTION

This is the most important chapter in the methodology for the elementary child for it enables us to study the areas and the volumes.

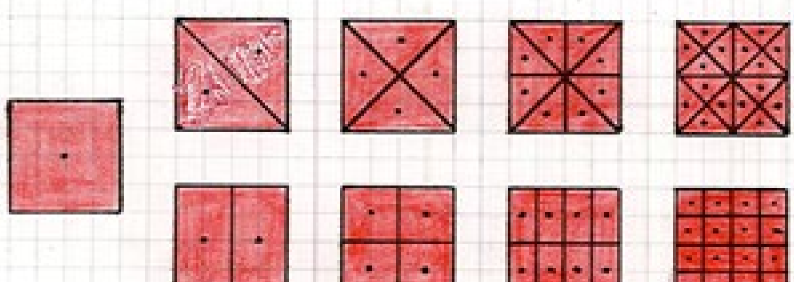
To give the child the concepts of congruence, similarity and equivalence, it is not necessary to wait until he has finished the program to this point. It IS necessary that he be able to recognize, without doubt, ALL the figures. Therefore, the best age to begin is at about 8 years. That is, more or less, after the study of the angles. The precautious presentation of this chapter is necessary because the experiences done with congruence, similarity and equivalence require a long time---in such a way that, if we start very early, we will give the child the possibility to work with his hands and thus he does not receive only the lessons that the teacher gives. The succession of presentations of the concepts is the same: the first presentation at the sensorial level, the aim of which is to give to the child an idea ofwhat we want to present. Then, immediately afterwards, we give a language lesson which gives the exact meaning of the term. Then we have the application of the acquired concept which is done with the specific material called the Second Series of Constructive Triangles.

The last part of the work will consist in the comparative study of equivalent figures. That is, the consciousness of the relationship between lines and equivalent figures. We will refer to some of the geometry charts. Finally, there will be a study of certain theorems on equivalence which takes us directly to the calculation of the area of plane figures.

Presentation: An Introduction to the Materials: the 9 metal insets of the square AND the 4 metal insets of the circle.

NOTE: The child should have, at this point, worked with the fractions, at least at a sensorial level so that he can pass easily from the identification of the circle as a whole to the square as a whole.

1. Present the 9 metal insets of the square divided in two modes and display like this:



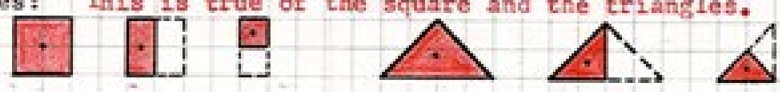
CONGRUENCE, SIMILARITY, EQUIVALENCE. . .
Presentation: Introduction of the Materials. . .

2. Note the constuction of the various fractions of the square.
   a) The whole: This is a square. All of the figures in these frames are squares, but they are divided in different ways.
   b) Top row, #1: This is a square divided into two parts by the diagonal.
   c) Bottom row, #1: This is a square divided into two parts by uniting the mid-points of the opposite sides.
   d) Top row, #2: This square has been divided by two diagonals, resulting in four triangles.
   e) Bottom row #2: This square has been divided by two lines uniting the midpoints of the opposite sides, resulting in four squares.
   f) Top row, #3: In this square we have two diagonals and also lines uniting the mid-points of the opposite sides, resulting in eight triangles.
   g) Bottom row, #3: Here we have divided each of the squares in the preceeding figure by joining the midpoints of one opposite pair of sides. The result is eight rectangles.
   h) Top row #4: Here we have divided each of the squares formed by uniting the mid-points of the opposite sides (as we did in the preceeding figure) by 2 diagonals, just as we divided the square first by a diagonal. The result is 16 triangles.
   i) Bottom row, #4: Now we have joined both of the midpoints of the opposite sides in each of the squares we formed in the second figure in this row.

3. Emphasize: that the diagonals always produce triangles and that the lines which unite the midpoints of the opposite sides always produce quadrilaterals of two kinds: either squares or rectangles.

4. Display one of the fractions from each of those figures on the bottom row. Here we see that the lines uniting the opposite sides produce: rectangle, square, rectangle, square.

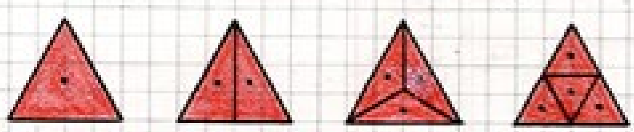
5. Emphasize: that each part of a figure is exactly one-half of the preceeding figure fraction or double the size of the succeeding one. Demonstrate this by pointing out a series: This is true of the square and the triangles.



NOTE: The material is made on the decimal system base. The whole square has a 10 cm. side. Then the first rectangle is 5 X 10 cm., second square is 5^2 cm., next rectangle is 5 X 2.5 cm., and the next square is 2.5 $cm.^2$.

6. Introduce the series of small fraction labels. The children match a label to each of the fractional parts.

7. Present the triangle insets in the same way. Here there are fewer insets.



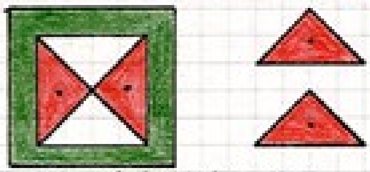
   a) To obtain the second figure, we trace the altitude which divides the figure into two equal parts.
   b) Then the thirds are formed by the bisectors of the angles, and each one is 1/3 of the whole.
   c) The fourths are formed by joining the midpoints of the three sides, and we form four equilateral triangles.

Presentation: FIRST LEVEL: THE CONCEPTS AND THE NAMES

Part I: The Concept and the Name of Congruent/Equal

Materials: The 9 metal square insets.

1. Give the concept of congruent. Take from the square inset divided by two diagonals the two opposite fourths. Superimpose the two so that the child sees that both parts cover each other perfectly.

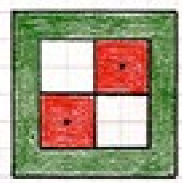


1. Let's imagine that this surface is formed by an infinite number of points. . . and this one, too.
Now we superimpose them.
Each point of the triangle below corresponds to each of the points of the triangle above.
Each figure covers the other figure perfectly.

2. The Language Lesson: giving the name.

2. These two figures have a special quality.
They are equal or congruent.
The word congruent is synonymous with t he work equal.
"Congruent" means "direct equality" or "superposition."

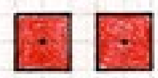
3. Repeat the presentation with other equal pieces: the fourths (as shown), the eighths, the sixteenths.



EXERCISE: #1. The teacher invites the child to bring two congruent parts. The child brings both parts and superimposes them. . .and we ask: Why are these two parts congruent?

#2. The teacher chooses one part and asks the child to bring the part equal to the one chosen.

#3. The child draws two congruent figures in his notebook and writes: These two squares are equal because each point of the surface of one directly corresponds to each point of the other one.



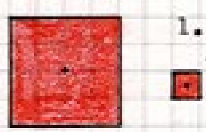
4. REPEAT THE EXPERIENCE OF CONGRUENCY WITH PARTS TAKEN FROM THE TRIANGLE INSETS.

Part II: The Concept and the Name of Similarity

Presented when the child has understood well the concept of equality/congruency.

1. Choose the whole square and the sixteenth obtained by the successive union of the mid-points of the opposite sides of the square. These are the two opposite parts in the set: the largest and the smallest squares.

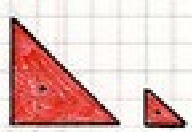
2. Give the concept of similar.



And the name.

1. What is this? A square.
What is this? A square.
They are both squares.
Are they equally big?
No---this one is bigger; this one smaller
They do not have the same value, the same area, the same weight. BUT they have the same shape. They have a special quality: they are similar.
What does similar mean? It means "to look alike" just as a boy resembles his father.
They have the same shape, but not the same size. If they did, they would cover one another.

3. Then repeat the presentation, using other parts: the rectangles of the ½ and the 1/8. The triangle ½ and the 1/16.



EXERCISE: Using the general pattern of those exercises presented for equality, ask the children to identify the concepts of similarity and equality.

4. REPEAT THE EXPERIENCE OF SIMILARITY WITH PARTS TAKEN FROM THE TRIANGLE INSETS:
The whole and the ½---both equilateral triangles.

EXERCISE: The child shows all the squares: the whole, the ¼, the 1/16, and he writes: These squares are similar because they do not have the same value nor the same size, but they have the same shape. We ask then: Are these two similar? these two? these two? We have discovered that all three are similar.

OR

The child takes the four right-angled isosceles triangles, each one-half of the previous one from the insets which have been divided by diagonals, repeating the same analysis of the similarity between all four and writing the conclusion.

Part III: The Concept and the Name of Equivalence

THE AIM: is to show that figures are equivalent when they have the same value (weight or area); but different shapes. This is the opposite of the previous concept.

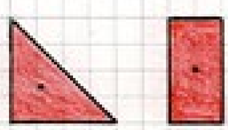
1. Giving the concept: using the whole square and the two insets of halves---that one formed by the diagonal and that one formed by the line joining the midpoints of the opposite sides; exchange the places of each, placing the two halves in the frame of the whole and then in the frame of the other halves; repeating the experience with the other halves and the whole.

1. We want to demonstrate that these are all equal---that the insets in each of these three square frames can go into the same frame.

2. Review the concepts of "fraction" and "½."

2. Here we have taken the whole, traced the diagonal and broken it into two parts---the result is two equal parts; one-half each.
We call each of the equal parts resulting from this division a fraction.

3. Take the "half" triangle and the "half" rectangle; note their equal value. Give the name of equivalence and define.



3. Let's identify these shapes:
What is this? A rectangle.
What is this? A triangle.
The two figures have different shapes and different names.
What is the value of this triangle?
What is the value of this rectangle?
Both are ½ of the whole.
Therefore they have the same value.
These two figures have the quality of equivalence.
Figures are equivalent when they have the same value, but different shapes.

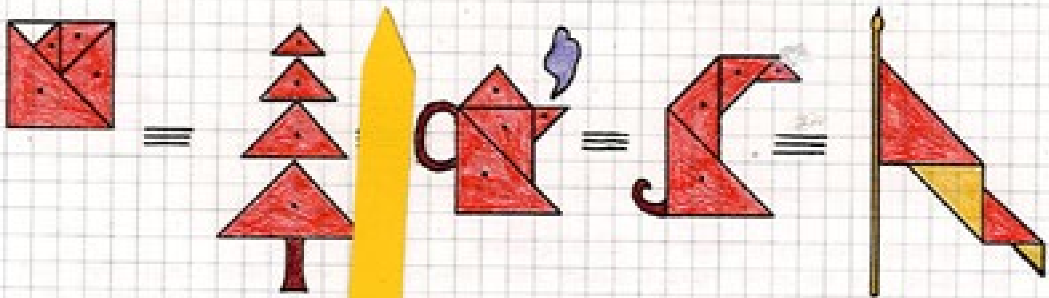
AT THIS POINT, WE CAN ORGANIZE A CHART WITH THE CHILD IN ORDER TO GIVE THE SYMBOLS:

No.	Concept	Equal Shape	Equal value	Symbols old	Symbols New	Conclusion
1	Congruent	yes	yes	$=$	$\cong$	The new symbol, then, means that congruent
2	Similar	yes	no	$\sim$	$\sim$	figures are, at the same time, similar
3	Equivalente	no	yes	$\equiv$ or $=$	$=$	and equivalent proven here to be true.

Note: The new symbols given seem more exact and more correct; the important thing is to introduce those that we will use and then to use them consistently.

CONGRUENCE, SIMILARITY, EQUIVALENCE. . .
Presentation: Part III: Equivalence. . .

EXERCISES: Using the metal insets, the child takes a particular group of the insets, fits them into the empty square frame to note exactly what value he is working with and then discovers figures and designs which are equivalent. Each time he forms a figure with that group of insets, he may replace them in the frame, noting the equivalence to the original group.



a) The child here has chos[...] the triangles formed by the diagonals dividing the square: the $\frac{1}{4}$, the $\frac{1}{2}$, the 1/8, [...] the 1/16.
b) He first fits them all [...]o the square frame, noting that almost the entire frame is filled. Only 1/16 [...]issing.
c) He explores the possib[...]ies, first finding the pine tree. . .and then trying to transform that pine tr[...] into another equivalent figure. He discovers the teapot, the animal, the flag. [...]h is equivalent to the other.
d) He proves this by putt[...] all the parts again into the frame, and it always fills the frame with the lac[...]g 1/16.
e) Now he cuts out the fi[...]es, using the metal insets a[...]s; and pastes the various equivalent figures in [...] notebook, writing: The pi[...] equivalent to the flag.

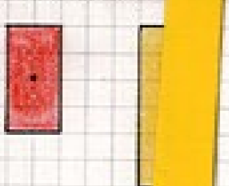
NOTE: Other ideas: the [...]res and the rectangles [...]ne.
the [...]ngles and [...]
the [...]
There is an infini[...] [...]and we must let the child exercise [...] crea[...]

Presentation: SECOND LE[...] The Fu[...]uivalence
These three concepts [...] been [...] Now we must define them more precisely.

Part I: The Concept of [...]larity: A [...]

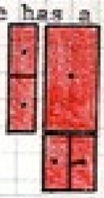
1. Take the $\frac{1}{2}$ rectangle [...] the 1/8 rectangle, and establish their similarity.

2. Then take one of th[...] two and present another rectangle, [...]of paper that is not proportional in [...]pect to the sides with the first and [...]efore not similar.



2. What is this? and this? Both rectangles
They have the same shape.
They have the same name.
BUT what was valid for the first two rectangles is not valid for these two.
These two figures, although they have the same shape and the same name, are NOT similar.
THEIR SIDES ARE NOT RESPECTIVELY PROPORTIONAL.

3. Show the proportio[...]ty of the sides of the two inset rect[...]es, first identifying their height and b[...]. Then show two small rectangles below the larger and to the side to demonstrate that in each case the larger one has a side twice that of the smaller.

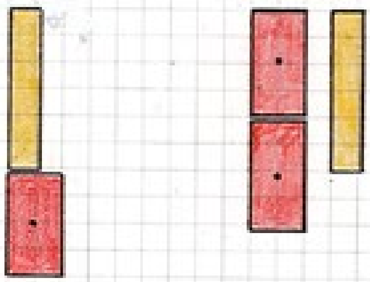


3. Let's look at our first two rectangles.
What is the base of the small one?
What is the height? Of the large one?
How many times larger is the large base than the small base?
The base of the larger is double the base of the smaller.
AND I can see that the height of the larger is two times that of the smaller.
Both the short and the long side of the large rectangle are respectively two times longer than those of the smaller.

CONGRUENCE, SIMILARITY, EQUIVALENCE. . .
Presentation: SECOND LEVEL: Part I: Similarity. . .

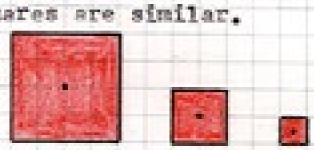
NOTE: These two rectangles are constructed in the same way; made from the same square; therefore, their sides are respectively proportional.

4. Take now the ½ rectangle and the other rectangle and make the comparison between the bases and heights. We see that we have an opposite ratio of dimensions. . . and prove that they are not similar.



4. Now let's compare our larger rectangle with this new rectangle.
What is the base? height? of the new?
First we compare the bases: we discover that the base of our new rectangle is just a little smaller than the base of the ½ rectangle.
Now let's measure the height of the ½ with the new rectangle height.
The height of the new rectmangle is nearly two times greater than the ½ rectangle.
So its base is smaller and its height is greater. . .than the respective base and height of the metal inset rectangle
We have an opposite ratio of dimensions.
SO---these two rectangles are not similar because the ratio between the two bases is not equal to the ratio between the two altitudes.

5. Show the three squares from the insets: the whole, the ¼ and the 1/16. Note that all squares are similar.

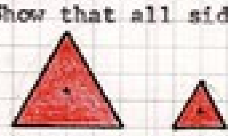


5. These three figures are similar because they are all squares and that is enough.
In order for squares to be similar, it is enough that they aresquares. WHY?
What do we know about the sides of the square?

6. ALL SQUARES ARE SIMILAR. AND ALL CIRCLES. AND ALL CUBES. AND ALL SPHERES.

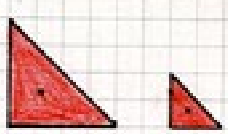
7. Considering now the triangles, a special problem:

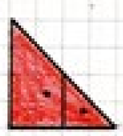
a) Take the two equilateral triangles from the triangle insets---the whole and the fourth. Show that all sides are proportional.

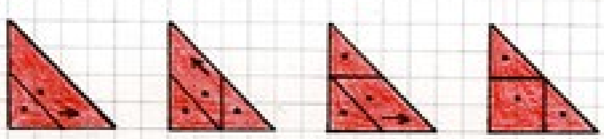


a) All equilateral triangles are similar becamse they are equiangular and all the sides are respectively proportional.

b) Show the ½ square obtained by the diagonal (right-angled isosceles triangle) and the corresponding 1/8. Identify the triangles, first by size; then by side and angle.
AND REDEFINE SIMILARITY.







2:1 2:1 2:1

b) What is this? A big triangle.
What is this? A small triangle.
They are similar because they have the same shape, but not the same value.
NOW we must classify them according to their sides and their angles.
They are both right-angled isosceles triangles.
IN ORDER TO BE SIMILAR, TRIANGLES MUST HAVE THE SAME CLASSIFICATION ACCORDI NG TO THEIR SIDES AND THEIR ANGLES.
So these two triangles are not similar just because of their similar shape, but because they have the same angles and the same sides: we must place them in the same position to see that they have respective angles equal and their sides respectively in proportion.
SO this right angle must coincide with this one. This hypotenuse must coincide with this one.
We can superimpose the two to see this coincidence.
We can also check the proportionality of the three sides. It is 2:1.

CONGRUENCE, SIMILARITY, EQUIVALENCE. . .
Presentation: SECOND LEVEL: Further Study of Similarity. . .

AGE: SECOND LEVEL OF SIMILARITY: about 10

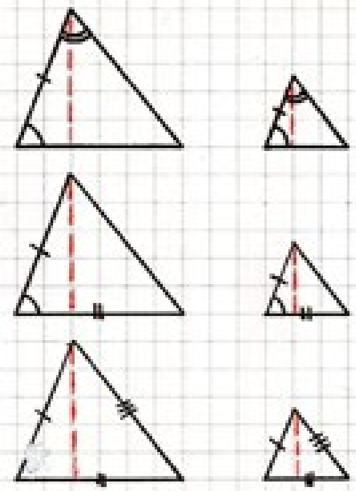
Presentation: THIRD LEVEL: Further Study of Similarity: The Criteria for the Similarity of Triangles

1. Consider the three cases in which we may establish similarity: use the acute-angled scalene triangle.

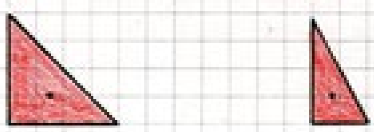
   CASE #1: There are two respective angles equal and the included side XXXXXXXXXX is proportional; therefore, the other angle will be equal and the other sides proportional, SO the two triangles are similar.

   CASE #2: Two sides are respectively proportional and the included angle is equal; therefore, the third sides are proportional and the other two angles equal. The two triangles are similar.

   CASE #3: The three sides of the triangles are respectively proportional; therefore the angles are equal. The two triangles are similar.

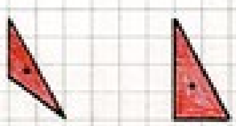


2. Consider the $\frac{1}{2}$-square triangle and the equilateral half from the triangle insets.



What is this? a triangle that is $\frac{1}{2}$ of the square.
What is this? a triangle that is $\frac{1}{2}$ of the equilateral.
Are they similar? NO
We mustanalyze them according to the sides and the angles.
Both have a right-angle, so both are right-angled triangles.
But one is isosceles and one is scalene.
Because we cannot identify them simultaneously according to the sides and the angles, they are NOT similar.

3. Consider the 1/3 triangle of the equilateral inset and the $\frac{1}{2}$ of that same triangle.



Let's classify each of these triangles.
One is a right-angled and the other is an obtuse-angled triangle.
One is scalene; the other isosceles.
These have two things against them---
THEY ARE NOT SIMILAR.
They are similar only in name.

Activity: The child constructs cardboard triangles; and determines whether they are similar or not. Here the child may, for example, construct two obtuse-angled triagles. He may determine that both are triangles, but that is not enough. They are both obtuse-angled triangles, but that is not enough. They are both scalene triangles, but still that is not enough. He discovers that BOTH OBTUSE ANGLES MUST BE THE SAME SIZE OR THE TWO TRIANGLES WILL NOT BE SIMILAR.

NOTE: It is through these experiences that the child discovers that in the triangle work, we must be much more precise when we consider the similarity of two figures.

Presentation: **Part II: The Further Study of Equivalence**

When we say that two figures do not have the same shape, but that they are equivalent, we indicate that they have a)the same area, b)the same weight, c) the same price, d)the same value.

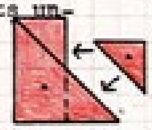
We have shown that the half of the square formed by united the midpoints of two opposite sides (rectangle) and the half of the square formed by tracing one diagonal (the triangle) are equivalent, but now we want to show equivalence more precisely.

1. Display both the rectangle and the triangle halves of the metal insets.

   Weigh the two if possible.

1. We know that these two figures are equivalent.
   If we weighted them, they would have the same weight.
   If they have the same weight and are made of the same material---as we see they are---they then would probably have the same price.
   AND if I made these two figures of cardboard, they should require the same amount of paper. Let's try it.

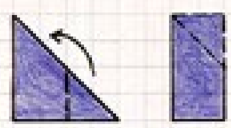
2. Show the common side, that side which is an equal length for the two. Putting those two sides together, superimpose the figures. . .then show how the 1/8 (triangle piece fits on both those parts uncovered.



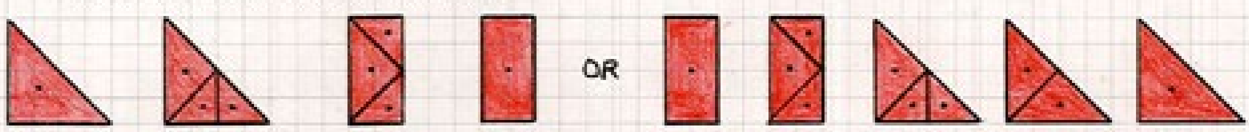
2. When I try to superimpose these two figures, I see that there is a part of each figure not covered.
   Is there a fractional part from the inset that corresponds to this space?
   YES---the 1/8 triangle fits both here and here.

3. Begin then with the cardboard figure. Cut, out of heavy paper, the 1/2 triangle. Then fold one of the catheti in half, back upon the figure and cut the piece. Show it as the top part of the new figure which is the rectangle.
   NOTE: If you begin with the rectangle, transforming it into the triangle, it is necessary first to fold the rectangle into a square---in half---then fold the square in half and cut.

3. Now we can do the same process with the paper.
   I want to transform this triangle into this rectangle.



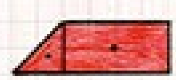
4. **EXERCISE:** The child begins with the metal inset of the 1/2 triangle; and, by using other inset parts, transforms that part into the rectangle. We begin by displaying the 1/2 triangle and at the other end of the mat, the right side, the 1/2 rectangle. The child must then discover the passage: here exchanging the 1/2 triangle for two 1/4s, then one 1/4 for two 1/8s.



EXERCISES: Of transformation. Showing equivalences between figures. Using the metal insets of the square as a bank, the child begins with two figures he knows to be equivalent. Then, through passages of exchanges, he seeks to arrive at the equivalent figure.

$\frac{1}{2}$ to $\frac{1}{2}$ || $\frac{1}{4}$ to $\frac{1}{4}$

Note: A more difficult variation: tape two inset parts together. The child seeks to discover the equivalence parts. This is a preparation for the equivalent area of plane figures.



Indirect Aim of equivalence work: to prepare the child for the next work: 1) the study of the relationship equivalent figures, 2) area, 3) detailed study of Pythagorean theorem.