FIRST LEVEL: Exploration of Figures. . .

The Constructive Triangles: First Series

Materials

The First Box of constructive triangles contains:

a) 2 yellow right-angled isosceles triangles; black lines along one of the equal sides
b) 2 green right-angled isosceles triangles; black line along the hyponteneusè.
c) 2 yellow right-angled scalene triangles; black line along the minor leg.
d) 2 grey right-angled scalene triangles; black line along the hypoteneuse.
e) 2 green right-angled scalene triangles; black line along the major leg.
f) 1 red right-angled scalene triangle; black line along the major leg.
g) 2 yellow equilateral triangle; black line along one side.
h) 1 red obtuse-angled isosceles triangle; black line along the longest side.
(This is the companion of the right-angled scalene red triangle)

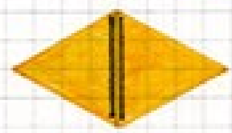
Presentation

1. Show the two equilateral triangles and ask the child to classify them, verifying that they are equal by superimposing them one on the other.

1. What is this triangle?
And this one?
Are they equal?
How shall we find out?

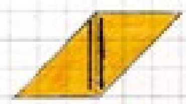
2. Point out the two black lines and unite the two triangles at those black lines. . .holding one in position and moving the other one to it. Identify the new figure. Have the child now repeat the positioning together.

2. Notice the black lines on each of the two triangles.
When we join the triangles on the side of the black lines, what new figure do we form?
This looks like a rhombus!!



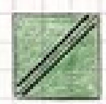
3. Proceed then with the two yellow isosceles and discover the common parallelogram.

3.



4. Then the green isosceles which when joined on the hypoteneuses make a square.

4.



5. The yellow scalene makes a parallelogram.

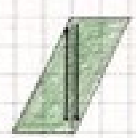
5.

6. The grey scalene, with black lines along the hypoteneuses, makes a rectangle.

6.

7. The green right-angled scalene makes another common parallel-ogram.

7.



8. Review the figures and what new figures have been formed with their combination.

8. The equilateral triangles have given us a rhombus.

9. Activities: the children cut out the figures of colored paper, then paste them in their new notebooks, writing the name of the new figure.

GEOMETRY COMMANDS

1. Plane Insets
   a) Using the demonstration tray, take the square from its frame.
   b) Then, one after the other, place all the insets of the geometry figures in the frame of the square.
   c) List all the figures that fit within the frame.

2. Plane Insets
   a) This line is 10 cm. long:

      Using the frame, draw a black pencil, the contours of the figures found in the first drawer.
   b) Then, one after the other, place on the red line that is on this command, the insets belonging to the frames that you have drawn.
   c) Finally, with a red pencil, draw on that figure, the position of the line equal to the red one.
   d) Do the same work with the figures in the 1st drawer.
   e) Do the same for the 2nd drawer.
   f) Do the same for the 3rd drawer.
   g) Do the same for the 4th drawer.
   h) Do the same for the 5th drawer.
   i) Do the same for the 6th drawer.

3. Plane Insets
   a) On your notebook, copy this table:

1 side	2 sides	3 sides	more than 4 sides	the sides are not possible to count

   b) Count the sides of each figure of all the drawers, and for each figure, write the name of the figure on the proper place.

4. Plane Insets
   a) Take these 8 insets: decagon, square, heptagon, circle, pentagon, equilateral triangle, nonagon, hexagon, octagon.
   b) Rotate each inset as you would the wheel of a bicycle.
   c) Line up the figures and draw the figures: Begin with those that rotate easily and finish with those that don't roll.
   d) Answer: Which roll better? Those with many sides or those with only a few sides.

6. Blue Constructive Triangles
   a) Take the equilateral triangles, join them and on a piece of paper, mark the contour of the quadrilateral that you have constructed.
   b) Do the same with the scalene triangles.
   c) Do the same with the isosceles triangles.

7. Blue Constructive Triangles
   a) Answer: How many figures have you constructed using the isosceles triangles? Why?
   b) Answer: How many figures have you constructed using the equilateral triangles? Why?
   c) Answer: How many figures have you constructed using the scalene triangles? Why?

8. Constructive Triangles
   a) At the window, copy by tracing, on a sheet of paper, the design on this command.

   b) With a pencil, unite the points following the order of the numbers: 1-2-3-4-1.
   c) Joining two points, divide the figure into 2 parts.
   d) Answer: What figures are formed by the 2 parts you obtained?

9. The Pinwheel
   a) Construct a pinwheel with 12 points.
   b) Draw this pinwheel, following the contours of each triangle.

10. a) Construct a pinwheel with 4 points.
    b) Draw this pinwheel, following the contours of each triangle.

11. Fundamental Concepts
    a) List the things that can be called "solid."
    b) List the things that make you think of a surface.
    d) List the things that make you think of a line.
    e) List the things that make you think of a point.

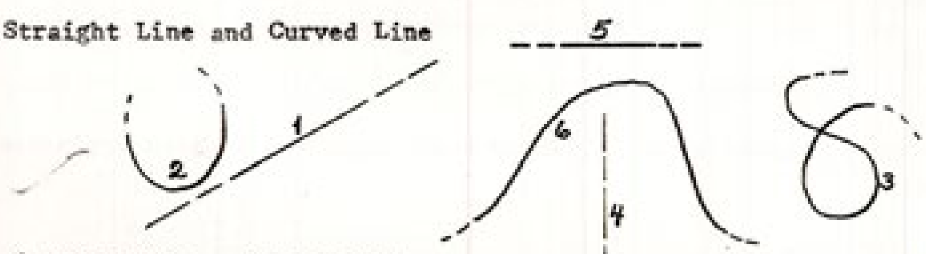
12. Fundamental Concepts
    a) Take a box of beads, take one of them: it is the image of a point.
    b) Then place one next to another, then place many other. . . . . . . .points.
    c) Answer: What have you constructed?

13. Fundamental Concepts
    a) From the box of 10-bars, take one bar: it is the image of a line.
    b) Then, place it next to many other. . . . .lines.
    c) Answer: What have you constructed?

14. Fundamental Concepts
    a) From a package of sheets for drawing metal insets, take one of the sheets: it has the image of a surface.
    b) Then, place the one on another, then many other. . . . .surfaces.
    c) Answer: what have you constructed?

16. Fundamental Concepts
    a) With the material of the decimal system:
    a) With the points: construct a line.
    b) With the lines: construct a surface.
    c) With the surfaces: construct a solid.

17. Straight Line and Curved Line



   a) Line 4 is a straight line:
      Answer: Which other lines are straight?
   b) Line 2 is a curved line:
      Answer: What other lines are curved?
   c) Copy this table and complete it:

Curved Lines	Straight Lines

GEOMETRY COMMANDS:

18. Straight Line and Curved Line
    a) With a green pencil draw a curved line and with a blue pencil draw a straight line.
    b) Answer: When is a line straight?
       When is it curved?

19. Straight Line and Curved Line
    a) Copy the words and above them draw the grammatical symbol:
       ---the line
       ---the straight line
       ---the curved line

20. Straight Line and Curved Line
    a) Make a list of objects in the room limited by straight lines.
    b) Make a list of objects in the room limited by curved lines.
    c) Make a list of the objects in the room limited by both straight and curved lines.

21. Straight Line and Curved Line
    a) Copy this table:

Figures with contours made up of straight lines	Figures with contours made up of curved lines

    b) One after the other, take all of the insets, touch the contours of each and decide if the contour is made up of straight lines or curved lines.
    c) Each time, write the name of the figure in the proper place.

23. The Straight Line and Its Parts
    a) Draw a straight line.
    b) Answer: Does a straight line have ends? Why?

24. The Straight Line and Its Parts
    a) First draw a straight line.
    b) Then, on it, construct a ray.
       Finally, construct a segment.
    c) Answer: What is the difference between a line segment and a ray?

25. The Straight Line and Its Parts
    a) Draw a ray and write the name of the point where it begins.
    b) Draw a segment of a straight line and write the name of its points of beginning and end.

26. Positions of Straight Lines
    Copy the words and draw the grammatical symbols:
    ---a curve
    ---a straight line
    ---a horizontal straight line
    ---a vertical straight line
    ---an oblique straight line

27. Position of a Straight Line
    a) List the parts of objects in the room, limited by a horizontal straight line.
    b) List the parts of objects in the room limited by a vertical straight line.
    c) List the parts of objects in the room, limited by oblique straight lines.

28. Positions of a Straight Line
    a) Write with a blue pencil, draw a vertical straight line.
       Then, draw a horizontal straight line, in red.
       Finally, draw an oblique straight line in violet.
    b) In your own words, try to write the definition of an oblique straight line.

30. Positions of a Straight Line
    a) On the horizontal plane, draw horizontal lines and vertical lines.
    b) On a vertical plane, draw horizontal lines and vertical lines.

31. Positions of a Straight Line
    a) Fix 2 sticks on a horizontal plane
    b) Fix 3 sticks on a vertical plane.

32. Two Lines
    a) Take two very sharp pencils: one red and one black and bind them together with a rubber band.
    b) On a sheet of paper; draw a curved road.
    c) On a little piece of paper, mark the distance between the 2 lines to see that they are always equidistant.
    d) Answer: When are 2 lines parallel.

33. Two Lines
    a) Draw an oblique straight line.
    b) Then draw one parallel to it.
    c) Repeat the work with a horizontal straight line.
    d) Repeat the work with a vertical straight line.
    e) Answer: When are 2 straight lines parallel?

34. Two Lines
    a) Draw, with a red pencil, 2 convergent arrows and with a green pencil, 2 divergent arrows.
    b) Answer: When are 2 straight lines divergent?
    c) Answer: When are 2 straight lines convergent?

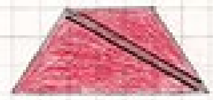
35. Two Lines
    a) Draw 4 right angles, using "the measuring angle."
    b) Cut out each right angle and join them together.
    c) Look at the sides of the 4 angles and then answer: when are 2 straight lines perpendicular?

36. Angles
    a) Draw a whole angle. Then fold it into 2 parts and then fold it once again.
    b) Answer: What type of angles have been formed with each fold?

37. Angles
    a) Draw the square found in the 1st drawer of the cabinet of plane insets.
    b) Then, using the measuring angle, "classify each angle of the square." Each time write the name of the angle.
    c) Finally, write your conclusions on each of the kinds of angles of the figure examined.
    d) Do the same work with the triangle of the first drawer.

38. Angles
    a) Draw the geometric figures contained in the 2nd drawer of the cabinet of plane insets.
    b) Then, using the measuring angle, classify each angle of each figure. Each time, write the name of the angle.
    c) Finally, write your conclusions about the kinds of angles found in each figure examined.
    d) Do the same work with one of the rectangles found in the 3rd drawer.
    e) Do the same work with the figures in the 5th drawer.
    f) Do the same work with the figures in the 6th drawer.

39. Angle and its Parts
    a) Draw an angle: Color: the sides in red; the vertex in blue; the size of the angle in yellow.
    b) In your own words, try to write the definition of these 3 parts of the angle.



10. When the child has worked with these for awhile, introduce the 2 red triangles: the colors the same, the figures different. One is the scalene triangle, with the black line along the major leg; the second is the obtuse-angled isosceles triangle. Together, joined on the black lines, they form the trapezoid.

DIRECT AIM: To give to the child the concept of constructing plane figures. He has also discovered that by joining two figures of the same kind, he is able to create certain new figures.

The Constructive Triangles: First Series: Box #2

Materials

The second box contains:

a) 2 blue equilaterial triangles, no lines.
b) 2 blue isosceles triangles, no lines. (right-angled)
c) 2 right-angled isosceles triangles, no lines.
d) a loose pair: one right-angled scalene triangle and one obtuse-angled isosceles, both blue.

Presentation

1. Repeat the introduction of the figures as in presentation #1: classifying the two figures and noting what new figure they form.	1. What is this figure? (rhombus When we put two equilateral triangles what have we formed? You may put them together any way you want.
2. Holding one of the two equilateral triangles in place, slide the other around the periphery until another figure is formed. Child discovers here that with the equilateral triangles he always forms the same figure.	2. Now let's try to slide one figure along the periphery of the other to find out if we can form other figures. What have we formed here? It is still a rhombus. And again we obtain a rhombus We always obtain the same figure with these two equilateral triangles.
3. Take the right-angled isosceles triangle and repeat the sliding process to discover the parallelogram and the square. It appears that there may be two parallelograms.	3. Now we want to form as many quadrilaterals as possible with these two triangles. What are they? Put them together however youw ant. What figure have you formed? Now we slide one around the contour of the other. I have formed a parallelogram, a square, and many another parallelogram. It's hard to tell.
4. Repeat with the right-angled scalene.	4. We form a parallelogram, a rectangle, and another common parallelogram. We see that these parallelograms are a little different from each other.
5. Pointing out the characteristics of each group of figures and those new ones they formed, state that we can form as many quadrilaterals as there are different measurements of sides.	5. How many different measurements do we have for the sides of this equilateral triangle? We have only one measurement because this side is equal to this one and this one How many different quadrilaterals did I form? One length for the sides---one quadrilateral.

5. . .Drawing conclusions.

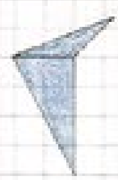
Show the two parallelograms formed with the isosceles:
Then with chalk, draw the periphery of one and discover that the second one does not fit unless it is turned over: that it is the mirror image of the first.

5. With the isosceles triangle, there is one length for the two equal sides and one other length.
How many quadrilaterals did I form?
2 or 3---I'm not sure.
This one looks like this one, but something is different.
If I have a triangle with two different lengths, I should be able to construct just two quadrilaterals.
It seems I have constructed 3.
Then let's try to fit the second parallelogram into the silhouette of the first.
I can't superimpose them this way, but I must be able to superimpose them if they are equal.
I must turn this second one upside down.
It is the mirror image of the first one.
So we have constructed only one parallelogram and one square.

With the scalene triangle, we have three different lengths---and we can form three different quadrilaterals.

6. Review the figures, how many lengths each has and the quadrilaterals formed.

7. Take the last two triangles, the right-angled scalene and the obtuse-angled isosceles and repeat the periphery process of sliding. The child discovers the trapezoid and the "concave quadrilateral:"



The Constructive Triangles: First Series: Box #2 Variations

When the child has worked with these triangles, constructing quadrilaterials, for a period of time, invite him to discover variations with the material. . . to construct other quadrilaterals OR new triangles with the same triangles he has been using. Now he may turn one over in order to discover these new figures.

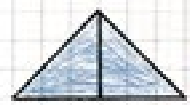
Presentation

1. Begin with the two equilateral triangles. The child discovers that by turning one upside down, he is still able to construct only one figure---the rhombus.

1. We know that with these two equilateral triangles we can make only one quadrilateral---the rhombus.
Now let's try to find new figures with these two triangles.
You may turn one or two or them upside down.
Still we can construct only one rhombus.

2. Then the two right-angled isosceles.

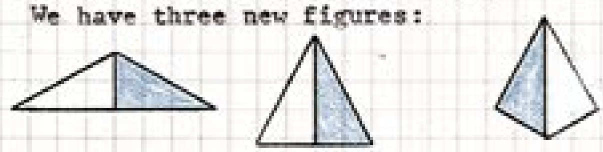
2. These two triangles formed a parallelogram and a square.
Let's see if it is possible to construct other quadrilaterals by putting one upside down. No.
Maybe we can construct other triangles.



So this new triangle is equivalent to the big parallelogram.
If we turn one triangle upside down, the figure is the same.

3. The right-angled scalene triangles.

3. With these two, we formed a rectangle a common parallelogram and another common parallelogram.
Now let's turn one upside down.
We have three new figures:



NOTE: The construction of the deltoid rhombus is important for future work in equivalences. It has perpendicular diagonals (like a rhombus), but unequal sides, formed in such a way that only one of the two diagonals is bisected. One diagonal divides the other into two parts that are equal, but the other does not, in turn, divide it equally. In the rhombus the diagonals cut each other in half. For one rhombus, there is an infinite number of deltoids, the rhombus being the limit. The limit of a deltoid with equal diagonals is the square.

Two triangles and a kite.
Maybe it is better to learn the correct name for this figure. . .the one that mathematicians use.
This is a "deltoid rhombus" or "deltoid."
The deltoid rhombus is a quadrilateral with diagonals perpendicular to each other, but those diagonals do not both divide each other into equal parts---only one diagonal is divided into equal halves.

4. With the last couple of triangles---the right-angled scalene and the obtuse-angled isosceles, we discover an important triangle: that one on which the area of the trapezoid is calculated.

4. Now let's take our last two triangles.
We have constructed with them a trapezoid and a concave quadrilateral.
Now we can form this triangle:

If we turn one over, we have nothing new.

The Constructive Triangles: First Series: Box #2 Variations. . .
Presentation. . .

NOTE: The work done with the first box of constructive triangles and the second box is important because it constitutes an indirect preparation for equivalences (and constructi on) of the plane figures. It is also an indirect preparation for the calculation of area.

5. Important Activity: The child cuts out the figures and pastes them as new figures.

6. Important: The child continues with the Geometry Commands.

The Constructive Triangles: First Series: Box #3

Material: A Rectangular box containing 12 equal right-angled scalene triangles, blue on one side and white on the other. All twelve triangles represent the left half of the equilateral triangle #2 , the special construction for which is shown in the figure opposite.

SIMPLE EXERCISES WITH A GREAT FUTURE: 3 Games: Let's Construct the Stars
3 Games: Let's Construct the Diaphragms

Presentation: 3 Games: Let's Construct the Stars

1. Show the box of new triangles, then take one triangle and analyze it according to sides and angles.	1. This is a scalene triangle. It is right-angled. It is a right-angled scalene triangle.
2. Take a second triangle, showing it on its right side to form then an equilaterial triangle.	2. We can form an equilateral triangle if we turn one of these triangles over and fit it with our first one.
3. Show that all the triangles are equal by putting the two halves together.	3. By dividing this equilateral triangle in half, we show that all the triangles are equal to one half of the equilateral triangle and equal to each other.
4. Give a simple nomenclature for the angles and the sides.	4. Let's call this the big angle; this the medium angle and this the small angle. And we will call this the long side, this the medium side and this the short side. Both these angles and sides have other special names.
5. Preparing for construction, set all of the triangles on the mat, resting on their hypotenueses and stacked like dominoes.	
6. With all 12, uniting the small angles at the center, construct the BIG STAR. Analyze the number of triangles and and the number of points in the figure.	6. This is the small angle. Let's construct a star by putting all the small angles together. How many points does this star have? How many triangles did we use?
7. Using the medium angle (60°), construct the MIDDLE STAR, and analyze.	7. This is the middle star, formed by putting together the medium angles. This star has six points---and we have used half of the triangles. So we can construct another one.

Equilateral triangle #1:

$s = 20$

$h = s\frac{\sqrt{3}}{2}$

$h = \frac{20\sqrt{3}}{2}$

$= 10\sqrt{3}$

$h = \sqrt{300}$

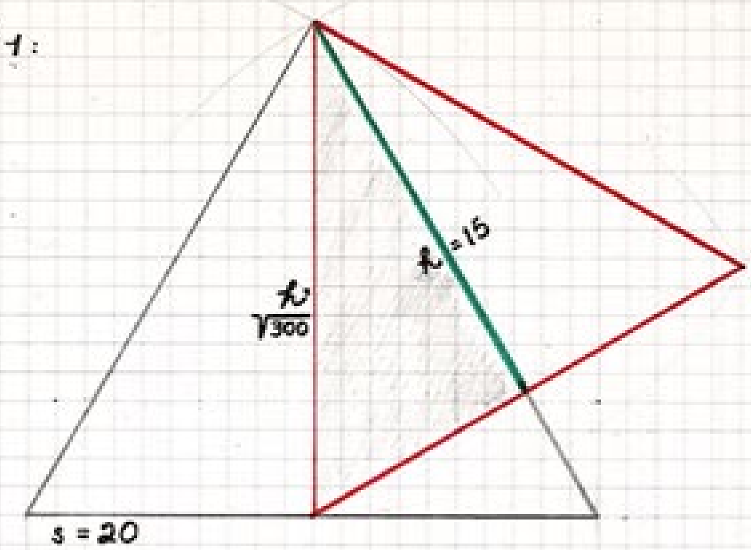
E. triangle # 2

$s = \sqrt{300}$

$h = \frac{s\sqrt{3}}{2}$

$h = \frac{\sqrt{300}\cdot\sqrt{3}}{2}$

$h = 15$



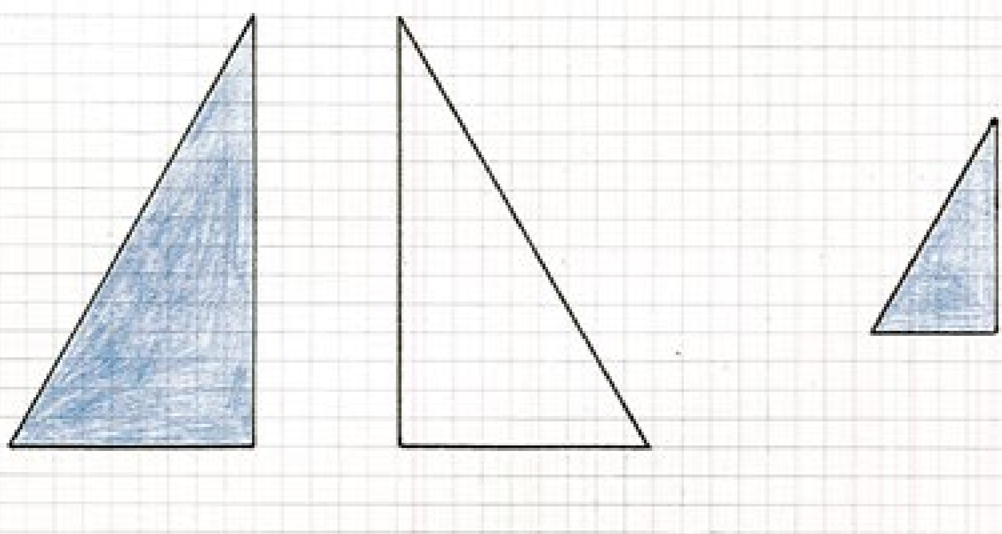
The left half of the equilateral triangle (shaded) has:

1) hypotenuse $= \sqrt{300}$

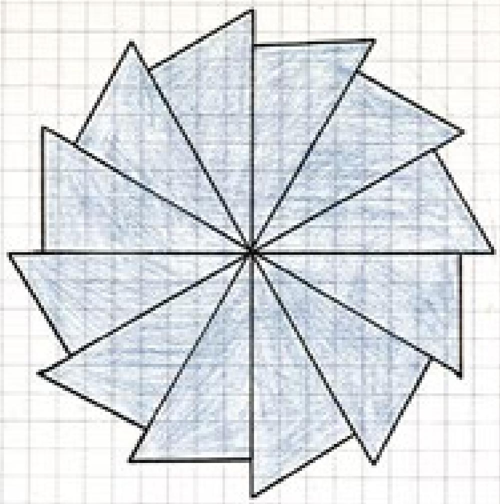
2) major leg $= 15$

3) minor leg $= \frac{\sqrt{300}}{2}$

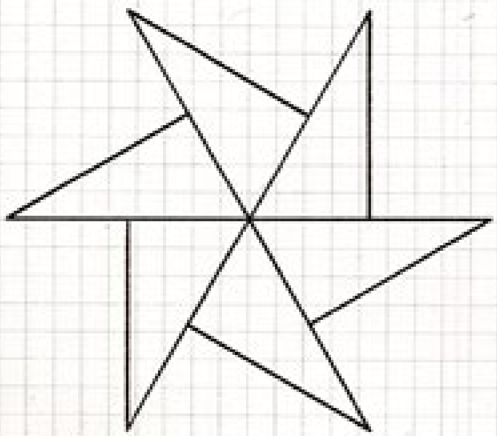
This is the triangle found 12 times in the third box of the constructive triangles: colored blue on this side shown and white on the reverse (showing the right half.)



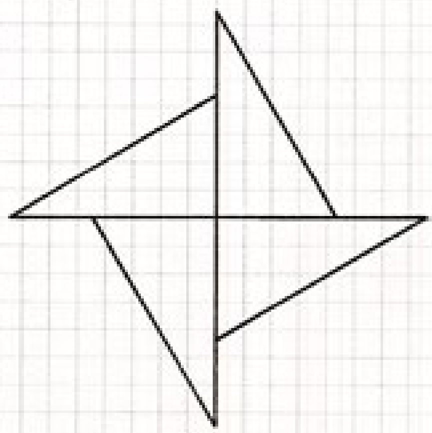
$= \frac{1}{2}$ shown for purposes of construction



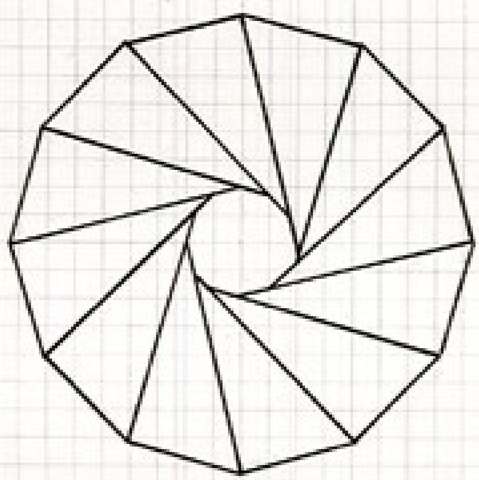
The Big Star



The Middle Star



The Small Star

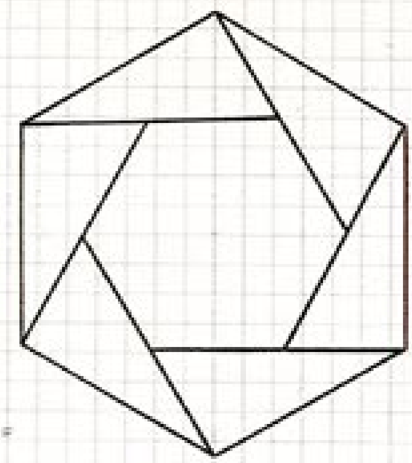


Construction =

1) inner circle diameter = ½ major leg

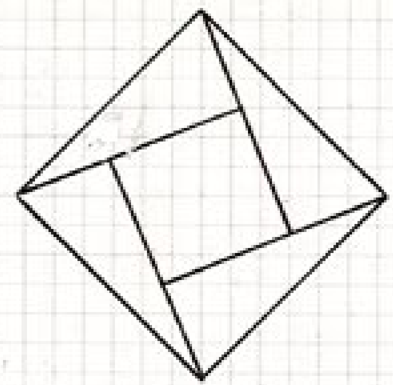
2) periphery circle = hypotenuse − 1.55mm.

Diaghragm of The Big Star



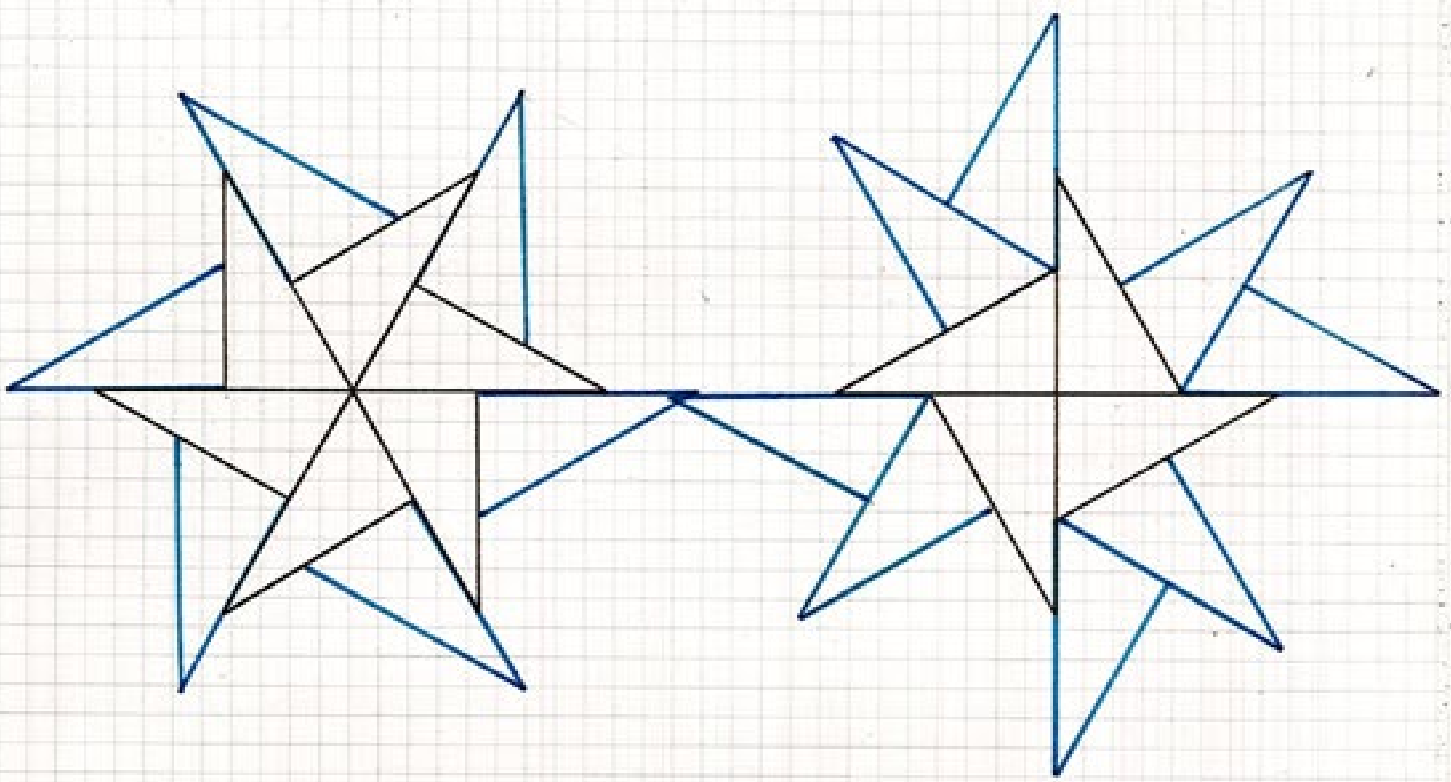
Construction :
1) interior circle = ½ major leg
2) periphery circle = major leg

Diaphragm of The Middle Star



Construction:
BAD
(center square much smaller)

Diaphragm of The Small Star



Two Variations on the Stars

The Constructive Triangles: First Series: Box #3. . .
Presentation. . .

7. Using the 90° angles at the center, construct the SMALL STAR. Analyze.

7. This star has four points; how many triangles did I use.
I can form, then, two more stars like this one with my triangles.

Presentation: 3 Games: Let's Construct the Diaphragms

1. Start with the first BIG STAR. By bringing all the exterior sides flush, form the diaphragm. Analyze: the opening in the center has a certain form: 12 sides; and the periphery also is formed of 12 sides.

2. Starting with the MIDDLE STAR, form the diaphragm in the same way.
Analyze: there are six sides on the figure formed at the center and six exterior sides.

3. From the SMALL STAR, form the diaphragm.
Analyze: the interior and exterior figures are formed of four equal sides. Two squares.
NOTE: From this demonstration we evolve the Pythagorean theory.

4. The child cuts out and pastes the stars and the diaphragms.

Presentation: Variations

1. Begin by forming the BIG STAR. Then take half of it and form the MIDDLE STAR. Add the second six, by the 90° angle, in the vertices of the star.

1. When we use all of the triangles in our big star, we can't make a variation.
With $\frac{1}{2}$ of these, the middle star was constructed.
Now I can add these other six in a special way to form a new star.
I have been able to use all 12 to form a particular star.
The inside star is made of 6 triangles---and the outside is composed of as many.

2. Begin with the SMALL STAR. Add the remaining 8 in couples (organized with both 60° fitted into the 120° angles of the small star's periphery.

2. Let's see where we can fit the remaining eight triangles.
In couples of 2, that is 4 couples, we can add them in a particular way to form this star.

PLANE GEOMETRY: LEVEL #2: The Knowledge of the Figures in Detail

THE CLASSIFIED NOMENCLATURES: Before being a work of geometry, the classified nomenclature is a work of reading. Therefore, the materials used are for the two levels: reading of words and reading of sentences. (labels for reading words; definitions for reading sentences.) The presentation of materials, then, should take this into consideration first.

Materials

The Classified Nomenclatures for Geometry are divided into 8 series, each marked with a letter A - H and each constituted of:

1) The folder containing: a) picture cards with no words
b) corresponding labels
c) definitions without the subject word
2) The wall chart, with the same picture cards and the subject written on each. Here the picture cards and the labels are united and are the CONTROL for the word done at the level of reading words.
3) The booklet: on the left side is the picture representation without the name; on the right is the subject and complete definition.
This is the CONTROL for the work done at the level of reading sentences.

THE CONSTRUCTIVE TRIANGLES IN ELEMENTARY SCHOOLS

When the idea of equivalence has been introduced to the children with the metal insets of a square divided into 2 to 16 equal parts, for further and more advanced work in the field of equivalences and of relative value between geometrical figures triangles are used. The material consists of three boxes: one of them triangular, the other two hexagonal.

T

The triangular box contains one large grey equilateral triangle the side of which measures 20 cm., and the same triangle divided into two, three and four equal parts (see fig. 1). The halves, each of wich is a right angled triangle, are coloured in green; the thirds, each of which is an obtuse-angled triangle, are yellow; and the fourths, each of which is an equilateral triangle, are coloured red. Each piece has one or more black lines to guide the child in combining the pieces so to reconstruct the large equilateral triangle. (Box and all pieces marked with yellow dots → for identification)

H_1

The basic element in the large hexagonal box is the obtuse-angled triangle which is equal to the thirds in the previous box. The contents of this hexagonal box are: one large yellow equilateral triangle of the same size as the one in the triangular box, with black lines along all three sides. Ten obtuse-angled triangles each 1/3 of the big equilateral triangle. Two of these are in red with black line along the largest side; two in grey with black line along one of the short sides; threee in yellow with black lines along both the short sides; three in yellow with black line along only the longest side. (All marked with light blue dot.)

H_2

The basic element in the smaller hexagonal box is the equilateral triangle equal to the fourths in the triangular box. There are eleven of them in this box. Six of them are coloured in grey with black lines drawn along two sides; three in green one of which has black lines drawn along two sides and the other two only along the base; two are in red with black lines only along one of the sides. Other contents of this same box are one equilateral triangle the side of which measures 17 cm., coloured in yellow, and two sets of the same triangle divided in three equal parts each forming an obtuse-angled triangle, coloured in red with black line drawn along the longest side.

These last mentioned pieces are itroduced at a later stage and not in the beginning.

The three boxes have the purpose of (a) rewiewing what had been experienced with the earlier box of Constructive Triangles; (b) extending this experience to geometrical figures with more than four sides; and (c) comparing the resulting figures in their various aspects; This last is done at a later period and in different steps/: first using one box at a time, then more boxes.

All this is done in the elementary school. But one can already introduce this material as for a sensorial exercise in the Children's House where children, like they did with the first box of Constructive Triangles, join the pieces along the black line and construct varoius figures. Each box is presented separately. Then when the same material is taken up in the elementary level, before being introduced to the new part, this part is done once again for some time so that the children may get re-acquainted with the material and recall their previous experience upon which the new part must necessarily be based.

The Constructive Triangles, Second Series

Notes and comments on "The Constructive Triangles in Elementary Schools", an article by Mario Montessori taken from the manuscripts of Maria Montessori.

Additional materials needed for the work:

1. Colored paper
2. Scissors
3. Scotch tape, glue.
4. Ruler, compass.

T: First box, triangular. The figures contained are proportionally twice those of the corresponding metal insets. Show this is the first introduction to the material.

The black lines on one or more sides of each of the parts of the triangle indicate how the whole has been cut.

The arrangement in the box is, from the bottom: the whole, 1/2s, 1/3s, 1/4s.

H_1: Contains 11 pieces. This is second box, large hexagonal.

H_2: Third box, small hexagonal. The central yellow triangle is that triangle constructed equilaterally from the altitude of the equilateral triangle side = 20 cm. (that triangle in the first two boxes). The side then, of this triangle = $\sqrt{300}$ or approximately 17 cm. It is the triangle, the half of which is seen in the third box of the first series of constructive triangles.

THE PRESENTATIONS: There are two levels of presentation for each box: first the sensorial level and then the detailed exploration which is the new part and the second level. The boxes are presented as listed: T_1, H_1, H_2 and then all three together.

A. Presentation: T_1

1. Ask the child to take the pieces from the box and to arrange them randomly on the mat. Then have him arrange them by color, stacking the equal parts in groups.

2. The child then forms the figures of each color group along the black lines. He must move the pieces directly towards one another---moving one or the other until the black lines join---but not sliding them up and down along those lines.

3. Review the fractional value of each of the fractional parts. (The child should have worked with the fractions inarithmetic before these experiences.)
   a) Superimpose the two halves on the whole, showing equivalence.
   b) Then superimpose the two halves one on the other.
   c) Give the name of the part.

3. This is the whole.
These two parts are equivalent to the whole.
They are also congruent to each other.
Each part is a half.

4. GAME. The child arranges the four triangles shown now in order, beginning with the whole. Then he names the triangles: This is the whole. This is the whole made of halves. This is the whole made of thirds. This is the whole made of fourths. THIRD PERIOD LESSON: What is the value of this part?

5. Bring the child's attention to the way in which each of the parts have been obtained from the division of the whole.

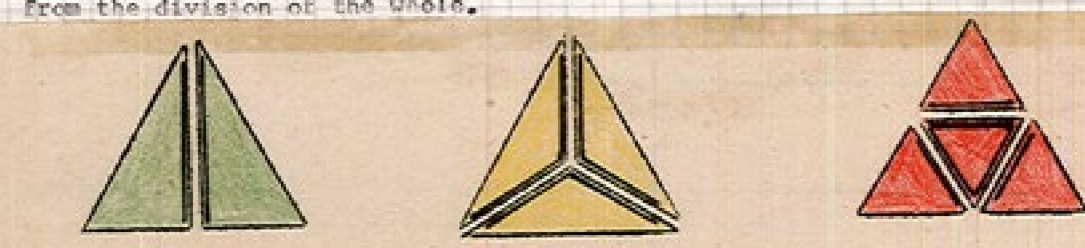


fig.1.

Of the three boxes the triangular is introduced first. Taking [illegible] the pieces first one asks the child to group them according to their colours and then join those of each group along the black line. Each group of pieces thus come to reconstruct the large grey triangle showing in this way that the large triangle has been divided into two, trhee and four equal parts.

Next step is to draw the attention on how this division has been done. By cutting the triangle along the perpendicular drawn from the vertex to it's base the triangle has been bisected. If one bisects all the angles in like manner, the lines meet at a point which is the center of the triangle. Cutting along these lines the triangle is divided into three equal parts. By joining the midpoints of the sides and cutting along those lines the triangle is divided into four equal parts (ref. fig. 1).

B After these constructions have been made the main element used at this stage is the right-angled triangle. So the thirds and the fourths are removed from the box and one combines the two halves in as many ways as possible to construct various figures and point out to the Child that each of them is equivalent to the large equilateral triangle (see fig. 2).* This in fact is a repetition of what the child has done with the right-angled scalene triangles of the first box of Constructive Triangles in the Children's House, and much of it may be remembered by the child. However, whereas previously the exercise was merely a sensorial impression, now the equivalence is a new light thrown on the same experience. C *Introduce transitive property (If $a = b$ and $c = b \rightarrow a = c$)

As parallel exercise children draw around the different figures and cut them out of coloured paper, each in a separate colour, showing the equivalence with the large equilateral triangle. Parallelly also the children are encouraged to use the compass, ruler etc. to draw the combination of figures.

A further step is to draw the attention of the child to examine the D. relationship of lines between (a) the large equilateral triangle and the various figures composed by combining the halves and (b) between the various figures themselves.

Thus, for example, between the large equilateral triangle and the rectangle formed by two halves: the long side of the rectangle is equal to the height of the triangle and the short side of the rectangle is equal to half the base of the triangle. The diagonal of the same rectangle is equal to one of the sides of the triangle (see fig. 3). This is an indirect preparation to understand the rules for finding the areas of different geometrical figures.

In the same way, are shown the relations between the rectangle and two parallelograms composed with the halves: drawing around the pieces so that the figures remain while being compared. The base of one of the parallelograms corresponds to the short side, and the height to the long side of the rectangle which can also be regarded as it's height (see fig. 4). The base of the other parallelogram corresponds to the long side and the height to the short side of the rectangle(see fig.5). Likewise one can also study relationship between the parallelograms themselves and between these and the big equilateral triangle.

A Passing now to the big hexagonal box: again, to start with, one asks the child to group the pieces according to their colour and then join them along the black lines. The resulting figures are: a hexagon, a rhombus, a triangle and a parallelogram. Then join two of the obtuse-angled triagles along the sides where there are no black lines one gets an arrow head. (see fig. 6).

Again some time is spent by the children in cutting out and composing the various figures in coloured paper, as well as in drawing similar ones on paper or in his drawing book.

B. 6. Remove the thirds and the fourths, thus isolating the halves---the right-angled scalene triangles---THE BASIC ELEMENT IN THEBOX. On the mat now are only the two halves and the whole.

7. Study the halves, summarizing the information the child already knows.

7. They are green.
They are two.
They are equal.
Each is equal to ½ of the grey triangles.

8. Now we proceedto construct with the two halves the greatest number of quadrilaterals and triangles possible without considering the black guide lines. We can turn over one of the triangles. We are able to construct: RECTANGLE, PARALLELOGRAM, A SECOND PARALLELOGRAM, OBTUSE-ANGLED ISOSCELES TRIANGLE, DELTOID. Five figures.

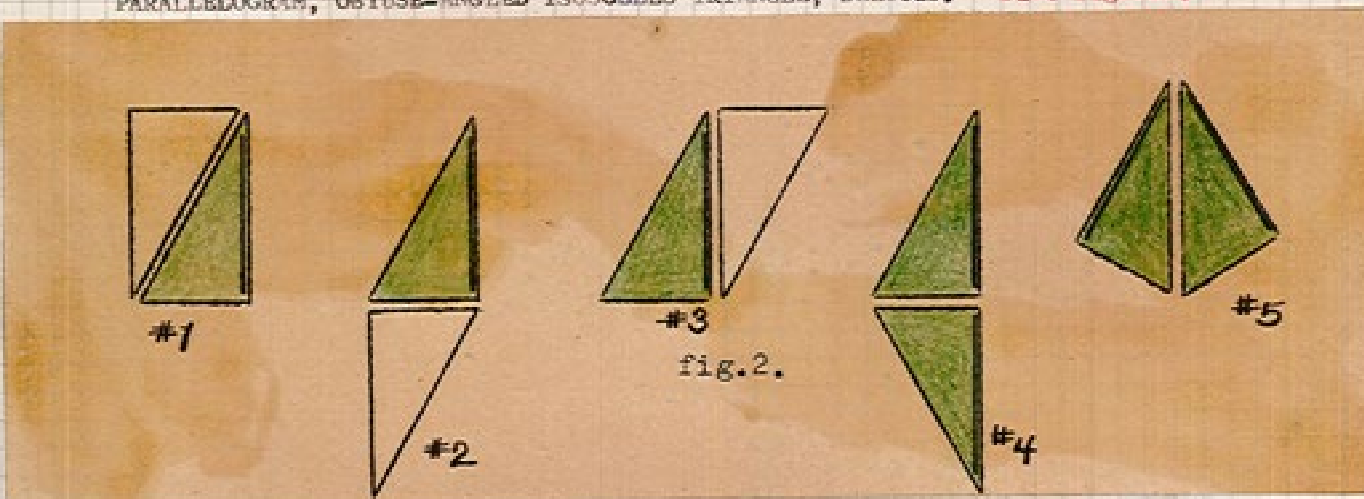


fig.2.

9. Invite the child to form the figures you have formed and to give their names. The child may explore all the ways of forming the figures, but one movement in particular is important: the <u>flip of one triangle</u>, where he must turn the triangle end to end is a movement he must make in both the construction of the rectangle and the parallelogram.
   a) To construct the rectangle, the child can move the right triangle or the left. He makes then two movements simultaneously. He moves the triangle to the opposite side and turns the top to the bottom part.
   b) In the parallelogram construction, he again may move either triangle. As he brings the left triangle to the right and down to the base of the right one, he again must flip the triangle over. OR he does the reverse.

10. Conclusion: We have formed 5 figures equivalent to the whole triangle.

C. Presentation #2: Equivalence Between One Figure and Many Other Figures

1. Superimpose the two halves on the whole to show congruency.

2. Form the first quadrilateral: the rectangle.

2. This rectangle is ~~equivalent~~ congruent to this triangle because it is formed by two equal parts of the triangle.

3. Repeat the experience with the second parallelogram.

3. This parallelogram is equivalent to the triangle because it is formed by two halves of the same triangle.

Each of the two figures is equivalent to the triangle because it is formed by respectively equal parts; therefore, the two figures in each case are equi-composed.

4. Examine each of the five constructed figures in this way, noting that figure's equivalence to the triangle. Each time the child declares that each figure is equivalent to the whole and gives the reasons.

5. Refer to the geometry charts #9 and #11 on which these same figures are shown.

THE SECOND SERIES OF CONSTRUCTIVE TRIANGLES. . .

T_1. . .

Presentation #3: The Transitive Property of Equivalence

1. Superimpose the two halves on the whole.

2. Form the rectangle.
   Then form the parallelogram.

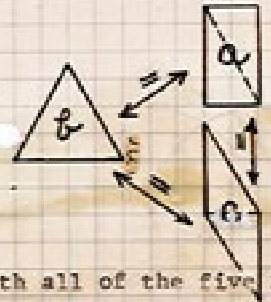
2. This is a rectangle.
It is equivalent to the whole because it is formed by two halves of the triangle.
This is a parallelogram. It is equivalent to the triangle because it is formed by the two halves of the same triangle.
The rectangle and the parallelogram, then, are equivalent to the same figure which is the whole triangle. Therefore, both figures are equivalent to each other.

NOTE: A useful scheme.

a) Reflexive property $a = a$

b) Symmetric property: If $a = b \Rightarrow b = a$

c) Transitive property: $a = b$ and $b = c \Rightarrow a = c$



$a = b$
and $c = b$
$\Rightarrow a = c$

($\Rightarrow$ a New Math symbol "implies that"; "therefore")

3. Demonstrate the transitive property with all of the five figures.

4. ACTIVITY: The child reproduces the five figures with colored paper, using a different color for each figure. He then puts them out to demonstrate the transitive property, showing the equivalence among the figures.

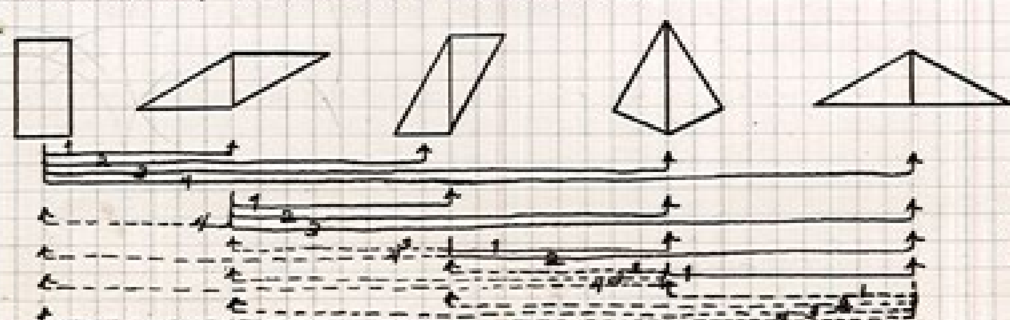
#2: Instead of using the figures as a guide, the child is encouraged to draw them using a compass and a ruler.

D. The Relationship Between Lines: IN TWO MODES

Now we turn to a consideration of the relationship between the lines of the equivalent figures discovered in the preceeding presentation. We may examine the relationship in two ways---each examination having two elements: the first term and the second term. The first term is that which provides a point of reference, that figure to which the others are compared. The second term of each comparison is the figure compared to the first.

In our first mode, then, the first term is the guide figure: the equilateral triangle. We examine the relationship of the lines between this triangle and the other five equivalent figures. (In our discussion and presentation, these figures will be described by number, as designated in fig. 2.)

In the second mode, we remove the whole grey triangle. We consider first the figure we constructed first with the two halves, the rectangle, as the first element in our study of the relationship between lines. And we discover the relationship between the lines of this first element and each of the other four constructed figures in succession. Then we repeat the process, but the first element becomes the parallelogram #2. In this series of comparisons, the first second element will be the rectangle; and we see that this comparison has already been made. And then, we we move to the third repetition of the exercise in which parallelogram #3 is the first element, the first two comparisons will be repetitions shown in the first two comparison series. In the final case, using the deltoid as the first element, we discover that each of the comparisons has already been made in a previous case.



...applying the property of symmetry. ...each figure compared to 4 others.

T. . .
The Relationship Between Lines. . .

Relationship Between Lines: Mode #1

Material
1. The whole triangle and the two halves which shall be used to show equivalence and also line relationships.
2. The figures of the five equivalent figures which the child has constructed in the activity of the previous presentation. AS WE BEGIN, REVIEW HOW THESE FIGURES WERE OBTAINED.

1. Begin with a restatement of the equality between the two halves and the equivalence between the triangle composed of two halves and the rectangle.

2. Show the whole triangle and the rectangle---REVIEW THE NOMENCLATURE.

2. This is the base of the triangle, this is the altitude, this is the side. . .side. . side.
This is the base of the rectangle, this is the altitude.
Why are these two equivalent?

3. By moving the halves of the triangle from the rectangle to the whole, we demonstrate the relationship of the lines: AN INDIRECT PREPARATION FOR THE CALCULATION OF THE AREA OFTHE FIGURES.

3. The altitude of the rectangle corresponds to the altitude of the triangle.
The base of the triangle is equal to twice the base of the rectangle.

4. Show the rectangle with the long base and repeat the exercise of comparison.

WE DO NOT HERE CONSIDER THE DIAGONAL RELATIONSHIP BECAUSE IT DOESN'T FIGURE IN THE CALCULATION OF AREA.

4. Now the base and the altitude of this rectangle have changed places.
Here the base of the rectangle is equal to the side or the base of the triangle.
The altitude is equal to $\frac{1}{2}$ the side of the triangle.


fig. 3.

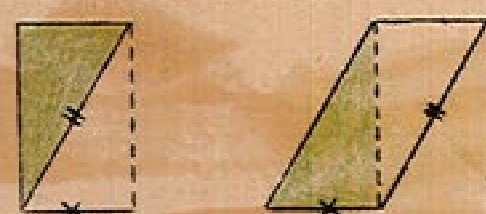


fig. 4.

5. Repeat the exercise with a comparison between parallelogram #2 and the whole triangle. Begin with a review of the nomenclature, using first the position of the parallelogram where the base is equal to altitude of the triangle---the minor or shorter base. Compare the relationship between the lines of the two figures. Then show the parallelogram in the second position, base being the longer side. Repeat the nomenclature identification and the comparison of lines.

6. Repeat the exercise with parallelogram #3 (fig. 3) and with the other two figures.

NOTE: At the conclusion of this sensorial experience, we can write statements of the relationship between the lines of the figures, as shown on Triangle Charts #12 and #13. Show those charts when the work is completed.

Relationship Between the Lines: Mode #2
Following the pattern described in the introduction to this study of the relationship between the lines, examine with the child the relationships between the lines of the five constructed equivalent figures. The whole triangle now is gone from this experience. The halves may be used as necessary to establish the relationships, but by superimposing the paper figures on one another, we provide an excellent sensorial experience of this relationship. (fig. 4)

H_1: The large Hexagon box.

Presentation

A. 1. Ask the child to put the pieces out and count them when he has arranged them by COLOR: 7 yellow triangles and 2 red triangles and 2 grey triangles.

2. Ask him to form as many figures as possible with the groups he has formed, uniting the figures along the black lines: he forms the equilateral triangle with three thirds, the hexagon with the large triangle plus three thirds, the rhombus with the two red triangles and the parallelogram with the two grey. HE NAMES EACH FIGURE.

3. Now we move to the construction of other figures, in which we can turn one piece over if necessary, and we ignore the black lines.

3. First take two of the yellow thirds to make this exploration:
We discover a rhombus equal to the red rhombus we constructed.
And a parallelogram equal to the grey.
And, by removing one third from the equilateral triangle, we have a concave quadrilateral we call an arrowhead.

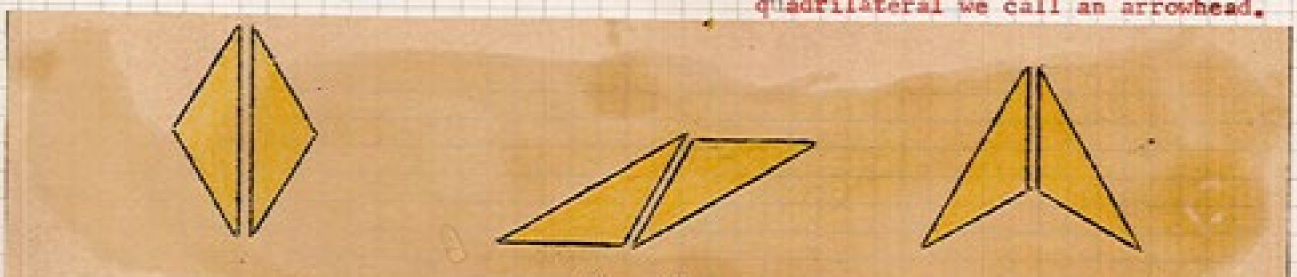
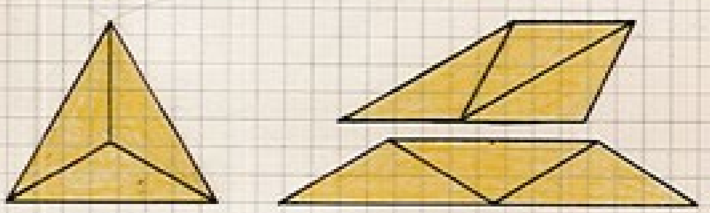


fig. 6



Then we work with three of the yellow thirds.
We discover the triangle equal to the guide equilateral triangle, an isosceles trapezoid, and an obtuse-angled trapezoid.

NOTE: We can compose various other figures with four of these triangles, beginning with the long parallelogram whose longer sides equal twice the longest side of the triangle. We DO NOT PROCEED WITH THIS EXPLORATION because using the four thirds brings us to 4/3, the improper fraction.

4. ACTIVITY: The child composes similar figures of colored paper.

5. Now the child separates the pieces according to shape. We note the congruency of the ten thirds. And the one whole triangle.

B. 6. Give each of the parts its fractional value, first superimposing three on the whole, verifying congrunce.

6. Each of the parts is 1/3 of the whole. We know that each of the three parts is congruent with the other.
And that all the ten are congruent with these three.
Then each of the ten triangles are 1/3 of the whole.

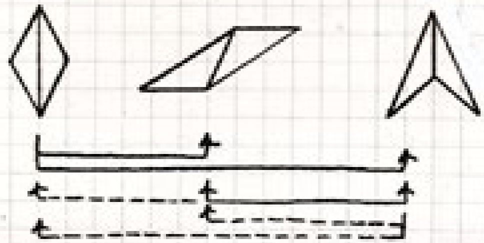
7. Construct each of the previously constructed figures, giving each a fractional value.

The rhombus is equal to 2/3 of the whole triangle.
The parallelogram is equal to 2/3 of the whole triangle.
The arrowhead, the concave quadrilateral, is equal to 2/3 of the whole triangle.
AND the isosceles trapezoid constructed with the three triangles is equal to 3/3 or one whole. And so the obtuse-angled trapezoid.

8. Now show the red rhombus, the grey parallelogram and the yellow concave quadrilateral, and display opposite the whole triangle. Restate the value relationship.

8. We know that each of these three figures is equal to 2/3 of this whole triangle.

9. Because the three are equal to 2/3 of the whole, we can say that they are equivalent among themselves. Remove the whole triangle and make the equivalence statements.



9. Each of these three figures is equal to 2/3 of the whole triangle. Therefore, each has the same value. And we can state that they are each equivalent to the other: The rhombus is equivalent to the parallelogram; the rhombus is equivalent to the arrowhead. The parallelogram is equivalent to the arrowhead; and, as we have stated in our first comparison is also equivalent to the rhombus. Therefore, we have stated that the arrowhead is equivalent to the rhombus and the parallelogram.

C. Presentation #2: Relationship of the Lines

MODE #1: The whole triangle is the first element; the relationship between its lines and the lines of the three constructed figures---rhombus, parallelogram and arrowhead, each successively the second element, is considered. (fig. 7 a, b)

MODE #2: Taking each of the three figures successively as the first term of comparison, the relationship of lines between it and the other two figures is considered. In this examination, the whole is not present. (fig. 7, c)

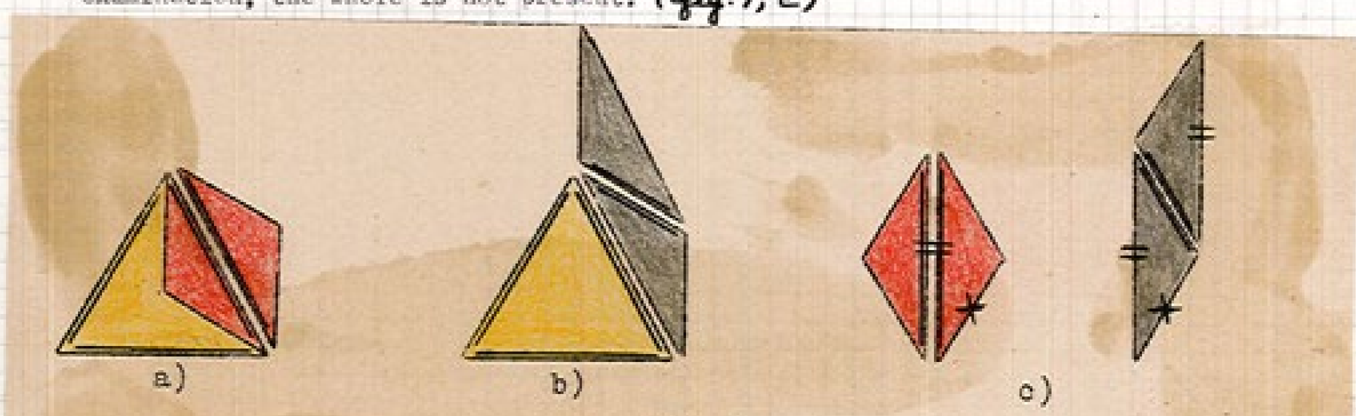


fig. 7.

D. Presentation #3: Relationship Between Two Figures of Different Fractional Value

1. Using now all the yellow triangles, form the hexagon with the equilateral triangle and the three thirds; then form the equilateral triangle with the remaining three thirds.

2. Using the sides of the equilateral triangle as a pivot, reverse the three smaller triangles upon that triangle, showing that the hexagon is equal to two equilateral triangles. . .which are congruent. fig. 8a

3. CONCLUSION: The hexagon is formed by two equilateral triangles.
AND the hexagon is the double of the equilateral triangle.
AND the equilateral triangle is one-half the hexagon.

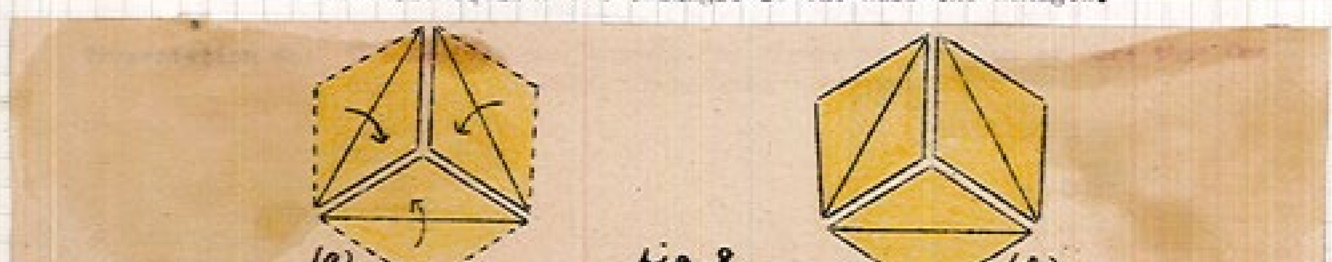


fig. 8

E. Presentation #3: The Equivalence Between One Figure and the Sum of More than One Figure Equal Among Themselves

1. With the yellow triangles, form the hexagon again and the equilateral triangle.

2. Remove the center triangle from the hexagon. There is an empty space. Place the three triangles which we have shown as the equilateral triangle into that place showing the congruency.

3. OBSERVE: The interior triangle is inscribed in the hexagon.
   The hexagon is circumscribed about the triangle.
   AND The vertices of the triangle coincide with the non-adjacent vertices of the hexagon.
   SO We can say, by joining these, we trace the inscribed triangle.

4. CONCLUSION: T_1 is inscribed in H_1.
   H_1 is circumscribed about T_1.

F. Presentation #4: The Relationship of Lines Between the Inscribed Triangle and the Circumscribed Hexagon

Here we consider: A. The hexagon with the equilateral triangle (the interior triangle.)
B. The hexagon with the rhombus.

WE ARE EXAMINING THE HEXAGON AS CONSTRUCTED IN THE PREVIOUS PRESENTATION.

1. Discovering the relationships of the lines in Case A. Note the meeting point of the bisectors.	1. Each side of the equilateral triangle (the interior triangle) corresponds to a particular line of the hexagon: the diagonal obtained by joining the non-consecutive vertices of the hexagon. The side of the hexagon is equal to a fractional part of the bisector.
2. (fig. 8b) Divide the hexagon into 3 rhombi. Case B is considered. Then reconstruct the hexagon.	2. The hexagon is equal to the three equal figures: the three rhombi. The major diagonal of the rhombus coincides with the diagonal of the hexagon obtained by joining the non-consecutive vertices. The side of the rhombus is equal to the side of the hexagon.
3. Compare the interior equilateral triangle with the rhombus.	3. The major diagonal is equal to the side of the equilateral triangle. The side is equal to a fractional part of the bisector.

H_2: Presentation #1: Introduction

1. Ask the child to put the contents of the small hexagonal box out, then to remove all but the large yellow triangle and the small equilateral triangles.

2. He now groups those displayed figures according to color: 6 grey triangles, 2 red, 3 green and the yellow equilateral.

3. The child forms the figures according to the black lines obtaining the hexagon, the rhombus, and the trapezoid. (A special trapezoid because the minor base is equal to each of the oblique sides and is equal to $\frac{1}{2}$ of the major base.)

4. Give to each constructed figure a fractional value. To do this the child counts the number of triangles first in the hexagon, then in the trapezoid, then in the rhombus.	I know that these pieces are all equal among themselves. The trapezoid is equal to three-sixths or one-half of the hexagon. The rhombus is equal to two-sixths or one-third of the hexagon. Then the hexagon will be three times the rhombus or six halves or 6/2.

II. Presentation #2: Relationship of the Fractional Value Among Figures Which do not Have the Same Fractional Value

Construct the trapezoid and the rhombus.

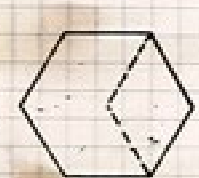
The trapezoid is 3/2 the rhombus.
The rhombus is 2/3 the trapezoid.

III. Presentation #3: Relationship of Lines

A. Between the hexagon and the trapezoid.
Move three top triangles slightly up to show the hexagon as divided in half.

A. If we divide the hexagon, we can say that the diagonal which joins these opposite vertices is equal to the major base of the trapezoid.

B. Between the hexagon and the rhombus.
Begin with the nomenclature of the rhombus. Then isolate one rhombus:



B. If we divide the hexagon in this way we seethat the long diagonal of the rhombus is equal to the diagonal formed by joining the non-adjacent vertices of the hexagon.
We know that the rhombi are formed of equilateral triangles.
SO the short diagonal of the rhombus corresponds to the side of the rhombus.
AND it corresponds to the side of the hexagon as well as one-half the diagonal.

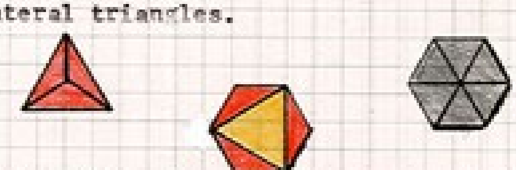
C. Between the rhombus and the trapezoid.

C. The short diagonal of the rhombus corresponds to the the oblique sides of the trapezoid.
The long diagonal of the rhombus is equal to twice the altitude of the trapezoid.

Presentation #4: Equivalence Between the Yellow Equilateral Triangle (T_2) and the Trapezoid

Material

1. The bi-colored hexagon formed of T_2 and three red thirds.
2. The grey hexagon, formed of the six equilateral triangles.
3. Three additional red thirds.
4. Three green equilateral triangles.



Presentation

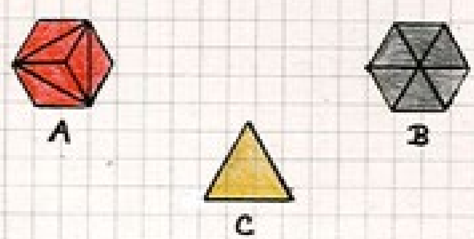
1. Form the bi-colored hexagon and the grey hexagon. Show the triangle composed of thirds.

2. Superimpose the bi-colored hexagon on the grey hexagon to verify congruency and then the grey on the bi-colored.

3. Remove the yellow interior triangle from the bi-colored hexagon and superimpose on the triangle which you have shown formed of the three red thirds. . .show congrepcy.

4. Substitude now in the empty space of the hexagon the three red thirds.

5. Show thusly, calling each by a letter.



5. The red hexagon is congruent to the grey hexagon.
We have seen that the red triangle is one-half of the first hexagon.
Since we know the grey hexagon is congruent to the first hexagon, then the yellow triangle T_1 is also equal to one-half of the grey hexagon.
Let's call the red hexagon A, the yellow triangle C and the grey hexagon B.

THEN: If $C = \frac{1}{2}A$ and $A = B$, $\Rightarrow C = \frac{1}{2}B$. We can write this relationship with the letters.

6. Remove the red hexagon. Remove $\frac{1}{2}$ of the grey hexagon.

6. The yellow triangle is equal to the grey trapezoid which is $\frac{1}{2}$ of B.

8.
After this one proceeds to pointing out value relationship first between each of the several above mentioned figures and the large equilateral triangle, and then among the figures themselves. For example, the rhombus is two thirds of the large triangle and so are the parallelograms and the arrow head (ref.fig. 6). Consequently it is deduced that the figures are equivalent among themselves. Here the fact is stressed again that two figures equivalent to a third are equivalent among themselves.

C.
With regard to relationship of lines it is found that the long diagonal of the rhombus is equal to the side of the large triangle (see fig. 7a) as are the long sides of the parallelogram (see fig. 7b); considering the relationship between the latter and the rhombus, two sides of the parallelogram are equal to the long diagonal and two other sides to the sides of the rhombus (see fig. 7c).

To form the hexagon the three yellow obtuse-angled triangles are used joining them by their long side to the three sides of the equilateral triangle. D. Then if one uses the latter as pivots and reverses the three triangles upon the equilateral triangle this is reproduced and one makes the child aware that the hexagon is equal to two equilateral triangles. Consequently one triangle is half of the hexagon (see fig. 8a). Subsequently another ascertainment is given. For this, one forms the hexagon again. E. Then by superimposing the other three yellow triangles on the equilateral triangle the hexagon appears now to be composed of three rhombuses. One also points out that the large equilateral triangle is inscribed in the hexagon (see fig. 8a).

It is not always necessary to attract the attention of the child to these facts. Often the child discovers them on his own and it is they who enthusiastically announce the discovery.

Like before F. the relationship of lines between the various figures is sought as well. For example, one can note that by joining the alternate vertices in a hexagon one obtains a triangle which is inscribed in the and it is the long diagonal of the rhombuses that form the sides of this triangle(see fig. 8b).

H2 I. Introduction

So far then with the contents of the large hexagonal box. A similar procedure is followed with those of the smaller hexagonal box. There are two steps in this. To proceed with the first, one removes the large equilateral triangle and the six obtuse-angled triangles. With what remains (all small equilateral triangles) one, as usual, first asks the child to group the pieces according to their colour and subsequently to join them along the black line. Thus one obtains a hexagon, a trapezium and a rhombus. II. Then for relative value between these figures, one points out that the trapezium is half of the hexagon and the rhombus, one third of it.

III.
With regard to lines one points out that in the trapezium the larger base is formed by two bases of the triangles that form it and the shorter by the base of one of them.

IV.
For the second step one brings into play the pieces which one had removed earlier from the box. By joining the obtuse-angled triangles along the black line one forms a hexagon. In it an inscribed equilateral triangle appears formed by the lines joining the vertices of the hexagon. One superimposes the yellow equilateral triangle on the one outlined in the middle of the hexagon. This shows that the two triangles are equal.

A Further exercise is to form another hexagon using the small equilateral triangles employed in the first step and then superimpose it on the previous one to show that the two are congruent. One then proceeds to explain that since the inscribed triangle is one half of the hexagon the value of the yellow one found in this box is equivalent to three small grey equilateral

THE PROBLEM: How can we explore the ratio between the two guide figures?

2. Take the six red obtuse-angled triangles included in this third box and constr the three rhombi which are a result of the black lines along the long sides of
WE KNOW THAT THE TRIANGLES ARE CONGRUENT AMONG THEMSELVES: THUS THE RHOMBI ARE CONGRU-ENT.

3. Compose the hexagon with these three rhombi. Now we have arranged these three rhombi in a hexagon which is equivalent to the three rhombi.

4. Replace the middle three thirds with the yellow whole. (this is the first time this yellow whole has been used.)
CONCLUSION: The yellow triangle inscribed in the hexagon is one-half of the hexagon because we have removed three of the six triangles.

5. REPEAT:

5. The triangle is inscribed in the hexagon and the hexagon is circumscribed about the triangle.

6. Take the six grey equilateral triangles and compose the hexagon.

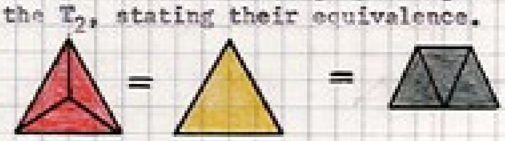
7. Superimpose the grey hexagon on the bi-colored hexagon. THEY ARE CONGRUENT SO WE COULD SUBSTITUTE ONE FOR THE OTHER.

8. Now show the all-red hexagon again AND the grey hexagon and the guide T_2.

Show one-half the grey hexagon by dividing it horizontally into two trapezoids, thus showing that part of the hexagon which is equal to the T_2.

8. We can say that this hexagon is exactly the same as this hexagon
AND
The yellow equilateral triangle is equal to one-half the first hexagon
AND
so it is equal to one-half the second hexagon. (grey)
SO
this triangle is equal to one-half of each of these hexagons.

9. Now show together the a)three thirds, b) the three-sixths (of the grey hexagon) and c) the T_2, stating their equivalence.

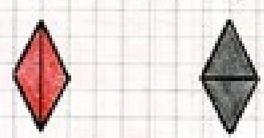


9. We have six triangles in each of the hexagons, so we can take three of the triangles of which the first hexagon is constructed OR three triangles of which the second hexagon is constructed---and each of those groups of three triangle is equal to this triangle T_2.

10. Identify the fractional value of each of the parts shown as groups of three.
TAKE ONE MORE GREY THIRD.

10. This is 1/3 of the triangle. . .and this is 1/3 and this. I have three thirds.
This is 1/3 of the triangle. . .and this is 1/3 and this. I have three thirds.
BUT I NEED ONE MORE.

11. Show the equivalence between the thirds.



12. Put the four grey triangles together to construct the large equilateral triangle.
SUPERIMPOSE T_1 to show the equivalence.

11. So how many grey triangles have I used to make up this triangle?
4

13. Show T_1 and T_2. CONCLUDE:

12. This triangle is equal to 3/4 of this triangle.
And this triangle is equal to 4/3 of this triangle.
Why are they made of different fractional units?
Because the wholes are different.

H_2. . .
Equivalence Between the Yellow Triangle and the Trapezoid. . .

7. Form the trapezoid with the three green equilateral triangles and superimpose them on the grey. Put the grey trapezoid aside. Verify the congruency. They are equal so we have left only the greey trapezoid. And the yellow triangle is equivalent to the green trapezoid.

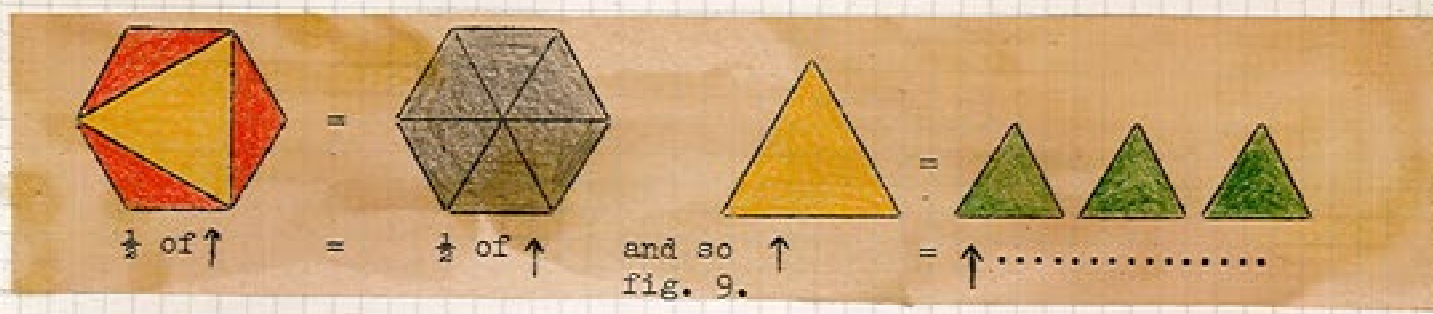


fig. 9.

IV. Presentation #5: The Ratio Between the Yellow Triangle (T_2) and the Grey Triangle (T_1) and Vice Versa

Material
1. T_1, grey; and the four red equilateral triangles from the first triangle box.
2. H_2, grey; the yellow triangle T_2 and the three green equilateral triangles.

Presentation

1. Form the first triangle (giving T_1) with the four equilateral red triangles; then superimpose that triangle on the grey guide triangle T_1 to verify congruency.

2. Now dispense with all but one red equilateral red triangle, shown together with the grey T_1.

3. Show also the yellow T_2 and the green trapezoid.

THE PROBLEM: What is the ratio between the grey whole and the yellow whole and vice versa?

NOTE: Here the red and the green equilateral triangles are used to show the passage and thus the relationship between the two wholes.

4. Consider the yellow triangle T_2 together with the trapezoid. Isolate one of the three equilateral triangles from the trapezoid.

4. We know that this yellow triangle is equivalent to the trapezoid. In fact, I need three of these figures to form the yellow triangle.

5. Superimpose the green equilateral triangle on the red, showing congruency.

5. What fraction of the grey triangle (T_1) is this red triangle? 1/4
What fraction of the yellow triangle (T_2) is this green triangle? 1/3

6. THE REASONING AND CONCLUSION: Then. . .the same piece has different fractional values, when we refer to different wholes. It takes 4 red triangles to make the grey figure. And it takes 3 green to make the yellow. SO. . .the yellow triangle (T_2) is 3/4 of the grey triangle (T_1). AND. . .the grey triangle (T_1) is 4/3 of the yellow (T_2).
THE YELLOW TRIANGLE IS TO THE GREY AS 3 IS TO 4;
THE GREY TRIANGLE IS TO THE YELLOW AS 4 IS TO 3.

Presentation #5: A Second Proof of the Above Ratio.

1. Display the yellow guide triangle (T_2) and the grey triangle (T_1).
Show also here the group of the six red thirds from the third box H_2.

H_2. . .

Presentation #5: THIRD PROOF of the Ratio Between T_1 and T_2

Materials

1. T_1 and T_2.
2. The red thirds from the box H_2.
3. The two green halves of the equilateral triangle from box T_1.

Presentation

1. Display T_1 and T_2.

2. Construct T_2 with the three red thirds, show congruency with the displayed yellow T_2 and put to the side.

3. Construct T_1 with the two green halves, show congruency with the grey triangle and put to the side.

4. Then, with the green halves, construct the deltoid.

5. Superimpose the three red thirds on the deltoid. Note the need for one more red third to cover the entire deltoid.

CONCLUDE: T_1 is equal to 4/3 of T_2. And, therefore, T_2 is equal to 3/4 of T_1.

I. THE UNION OF THE THREE BOXES: Presentation #1

1. Presentation of the four primary figures of consideration:
   a) Show the guide figure from box T_1, the grey triangle which is now T_1 and the yellow guide figure from box H_1. Verify congruency between the two.
   b) Show the large hexagon from H_1 formed of six yellow thirds from that box. (This hexagon, henceforth, called H_1)
   c) Show the guide figure from the third box H_2, the yellow triangle T_2.
   d) Show the grey hexagon formed of the six equilateral triangles from the third box. (This hexagon is now called H_2)
   e) Eliminate the large yellow triangle from H_1, working with the congruent grey figure now for color identification.

2. THE PROBLEM: What is the difference between the triangles T_1 and T_2? Repeat the prior conclusions.

   SHOW THE RED EQUILATERAL TRIANGLE ON TOP OF THE YELLOW FIGURE IN ONE HAND AND THE GREY IN THE OTHER, CONSIDERING WEIGHT.

2. We know that T_1 is 4/3 of T_2 and T_2 is 3/4 of T_1.
   So the difference is one red equilateral triangle (SHOW) which is 1/4 of T_1.
   If we put the 1/4 on the yellow triangle, together they would weigh the same as the grey triangle.

3. TWO PROOFS FOR THE CHILDREN: 1) Superimpose the yellow triangle on the grey. There is a grey frame around the yellow, a space not covered If we cut the grey frame and weighed it, it would weight what the 1/4 weighs. . .

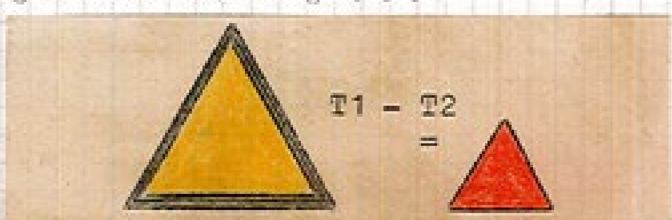
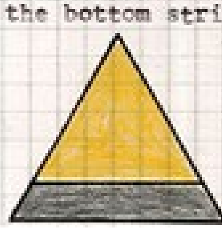


fig. 10.

2) Superimpose the yellow triangle with the top angle exactly corresponding to the top angle of the grey. Now it is the bottom strip which is equivalent to 1/4.



4. THE PROBLEM: The Ratio Between the hexagons: The terms are H_1 and H_2. The mediator to discover the ratio between the two is the whole grey triangle T_1.

5. Show the grey hexagon;. Analyze. and the grey triangle---T_1.

5. How many 4ths form the hexagon? (H_2) SO, if the hexagon is equal to 6/4, then the grey triangle is equal to 4/6 of the hexagon.

6. Show the yellow hexagon (H_1) and the grey triangle (T_1) and make the same analysis and comparison.

6. We know that H_1 is equal to two times T_1. How many fouths are there in H_2? ---6 NOW we can say, because H_1 is two times T_1, that there are 8 fourths in H_2.

7. State the ratio: The ratio between H_1 and H_2 is 8:6; that is, 4:3. . . just as the ratio of T_1 and T_2.

And the rationale: Why? Because we know that H_1 is the double of T_1
AND
that H_2 is the double of T_2.
SO the ratio is the same between the triangles and the hexagons.

II. Presentation #2: The Difference Between H_1 and H_2: Arithmetical

1. Show the two triangles: T_1 and T_2 and the red fourth, indicating this as the difference between the two triangles.

1. We know that $T_1 - T_2 = 1/4\ T_1$.
OR
$4/4 - 3/4 = 1/4$

2. We know that the hexagon is double its corresponding triangle. . .

So $H_1 = 2T_1$ and $H_2 = 2T_2$

Then $2T_1 - 2T_2 = 2 \cdot \frac{1}{4}$ or $\frac{T_1}{2}$

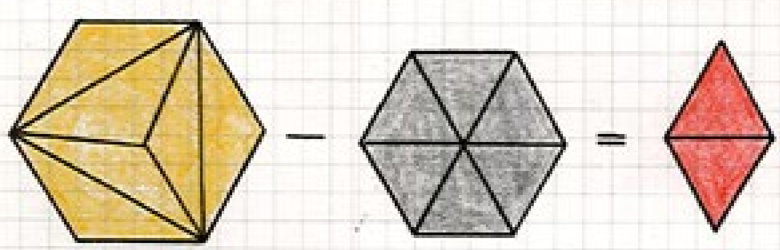
And $2 \cdot \frac{4}{4} - 2 \cdot \frac{3}{4} = 2 \cdot \frac{1}{4}$

$\frac{8}{4} - \frac{6}{4} = \frac{2}{4}$

And we can write:

$H_1 - H_2 = \frac{T_1}{2}$ (since we have no other hexagon to use in our comparison)

$\frac{8}{4} - \frac{6}{4} = \frac{2}{4}$



3. EXERCISE: The child cuts from different colored paper the two hexagons. He superimposes the smaller on the larger one, thus showing a frame. The value of the frame that results is equal to $T_1/2$.

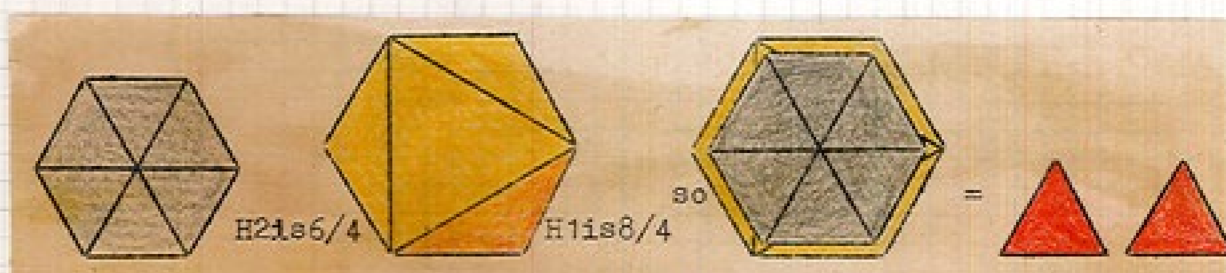


fig. 11.

A STUDY OF THE RELATIONSHIP BETWEEN T_1 and T_2. . .

Presentation #2: **2 Proofs of the Ratio 3:4**

Material: from T_1 the triangle T_1 divided into two equal parts (green halves) and T_1 from H_2 the triangle T_2, the red equilateral triangle, the red obtuse-angled triangle.

II. 1. With the red equilateral triangle and the red obtuse-angled triangle, repeat their equivalence (as in the previous presentation) and then unite the two so that one side of the equilateral triangle coincides with one of the equal legs of the obtuse-angled triangle. (fig. 12, last figure)

2. Take the corresponding green half of T_1 and superimpose it on the figure formed.

2. Both red triangles are 1/4 of T_1 so together they are equal to 2/4 or 1/ 2 of T_1.

3. CONCLUSIONS: To construct T_2 we need three of the red obtuse-angled triangles (1/3s). We know that we see that we need FOUR of these triangles to form T_1.
SO We have proved that T_1 is to T_2 as 4 is to 3.
T_1 is 4/3 of T_2
T_2 is 3/4 of T_1

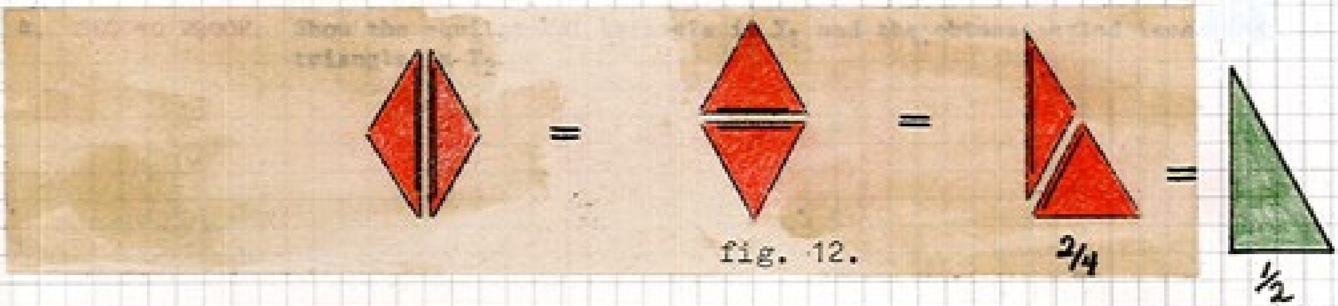
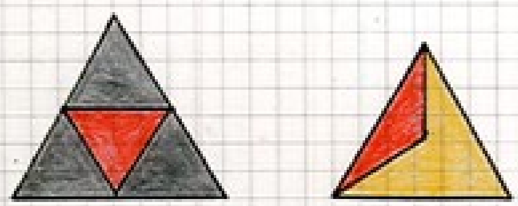


fig. 12.

4. SECOND PROOF: Show the equilateral triangle on T_1 and the obtuse-angled isosceles triangle on T_2:



We need four of the equilateral triangles to construct T_1; we need only 3 of the red obtuse-angled isosceles triangles to construct T_2. So the ratio, knowing that the two red triangles are equivalent, is:

T_1:T_2 is as 4:3 and
T_2:T_1 is as 3:4.

III. Presentation #3: **Using the Green Trapezoid to Make the Proof**

1. Construct the green isosceles trapezoid. (from box H_2)

2. Reasoning: Because each of the green equilateral triangles is congruent to the previous red equilateral triangle, we know that we need four of these green equilaterals to construct T_1. Therefore, THE TRAPEZOID IS 3/4 of T_1.

3. Show that it is impossible to construct T_2 with the equilateral triangles. Then SUBSTITUTE THE RED OBTUSE*ANGLED ISOSCELES TRIANGLES FOR THE THREE green equilaterals, noting that their equivalence has been proven.

4. Form T_2 with the three red thirds.

4. How many triangles did we need to construct T_1? How many to construct T_2?

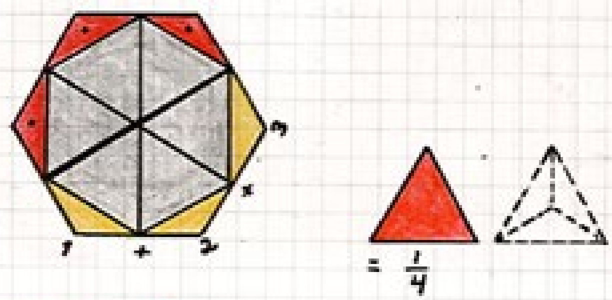
III. Presentation #3: The Sensorial Difference Between H_1 and H_2

Materials

1. The metal triangle inset divided into thirds.
2. H_1 and H_2 and the red rhombus, formed of two red equilateral triangles from box T_1.

Presentation

1. Superimpose H_2 on H_1, concentrically.
2. Rotate H_2 until the vertices touch the midpoints of the sides of H_1. . .now it is inscribed in H_1.
3. Superimpose the three thirds of the metal inset on the red equilateral fourth, verifying congruency and noting that one third is equal to one third of the equilateral triangle. And that all three together equal the equilateral fourth.
4. Impose the metal thirds on the yellow grame shown now around the grey hexagon, using the three in three of those spaces and concluding that now we have 1/4. Repeat the process, laying out a second fourth. The difference between the two is two fourths.



A STUDY OF THE RELATIONSHIP BETWEEN T_1 and T_2

I. Presentation #1: The Equivalence Between the Two Red Rhombi------the first formed of the equilateral triangles (1/4 T_1) and the second ob the obtuse-angled isosceles triangles (1/3 T_2)

Material

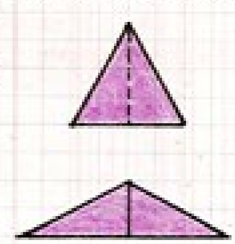
1. From H_2: The two equilateral red triangles and the two obtuse-angled isosceles red triangles.

Presentation

1. Form the two figures, using the black lines as guides. Name the two rhombi.
2. Superimpose the two to verify congruency. CONCLUSION: The Two are equal.
3. Note the construction of both from two equal pieces. Name one figure A and the second B. Then A = B. AND $\frac{1}{2}A = \frac{1}{2}B$.
4. Show the two halves of the rhombi. Note that the equilateral triangle is equivalent to the obtuse-angled isosceles triangle. (fig. 12)
5. Show the passage between the two using paper; showing the passage from either figure.
   a) Beginning with the equilateral triangle A, we fold along one of the altitudes and cut. Then we form the new obtuse-angled isosceles triangle by joining the two halves of the cut base. Superimpose on the wooden figure.

   OR

   b) Begin with the isosceles obtuse-angled triangle, cutting along the interior altitude and forming the new equilateral. Then superimposing it on the wooden figure.



triangles. And since each of the latter is equal to those which are 1/4 of the big equilateral triangle found in the triangular box, the value of this yellow equilateral triangle is 3/4 of the large equilateral triangle found in the triangular box. (see fig.9).

This reasoned explanation leads to comparisons, to other reasonings which involve first the two hexagonal boxes then also the triangular one. For convenience's sake let us call the big equilateral triangle contained in the triangular box and the hexagonal box, T1 and the big equilateral triangle of the small hexagonal box, T2. Also, let us call the hexagon made out of the pieces of the big hexagonal box, H1 and that made out of the pieces of small hexagonal box, H2.

I.
So the difference between T1 and T2 is one fourth of T1. Thus , if one surcimposes T2 on T1 a frame is determined the value of which would be equivalent to 1/4 of T1. (See fig; 10). In other words, again, T2 is 3:4 of T1.

II.
When the two hexagons are considered, the following reasoning is made/ as H2 is formed with six fourths of T1, this is its value. H1 is twice T1. And since T1 is 4/4 the value of H1 is double that, or 8/4. Consequentlyone can also deduce that the difference between H1 and H2 is 2/4 of T1. If one cuts on different coloured paper the outline ofthe two hexagons and superimposes them centrally the value of the resulting frame is equal to the difference of the two hexagons, that is 2/4 of T1 (see fig.11)

III.
It is possible to give also a sensorial proof of this difference between two hexagons. If the vertices of the smaller hexagon are made to coindide with the midpoints of the sides of the larger one, one finds that the smaller one comes to be perfectly inscribed in thelarger hexagon. Outside this inscribed hexagon there remains outlined six small obtuse-angled triangles. Cutting these off and joining them together they form two equilateral triangles each of which is equal to 1/4 of T1 (see fig. 11a° Thus H1 is shown to be H2 plus 2/4 T1.

I.
Now what about the relationship between T1 and T2? With the pieces out of the small hexagonal box, join two of the obtuse-angles triangles forming a rhombus and then two of the small equilateral triangles also forming a rhombus. Superimpose them and show they are congruent. So the obtuse-angled triangle is equivalent to the small equilateral triangle.

II.
In fact the two triangles are the half of the same rhombus; oneis derived by cutting the rhombus along its short diagonal; the other following the long diagonal. And indeed when joined, they form the right-angled triangle to be found in the triangular box which is half of T1; 1/4 + 1/4 = 1/2 (see fig. 12).

III.
Based on the above conclusion then one could arrive at others. For example, the trapezium made up of the small equilateral triangles is equivalent to T2 (see fig. 13) and that T2 is 3/4 of T1 (see fig. 14) - a conclusion which can be arrived at in different ways. IV The side of T2 correspond to the height ofT1 (see fig. 15).

One would think that all the above had been worked out before being given to the teachers and it was they who guided the children to the sucessive realizations. But to state facts as they took place , w at I now expose took a long time to be completed. Certain items were found in one country others in another. And many of these were discovered by the children themselves and shown to the teachers. Because the child, once his interest has been really aroused, uses the item as a key for further investigation and

5. State the ratio on the basis of the construction.

5. Then the ratio of T_1 to T_2 is as 4 is to 3. T_2 is 3/4 of T_1.

6. Note that the trapezoid is equivalent to T_2. Show the two figures.

6. And we said that the trapezoid is 3/4 of T_1 SO the trapezoid is equivalent to T_2.

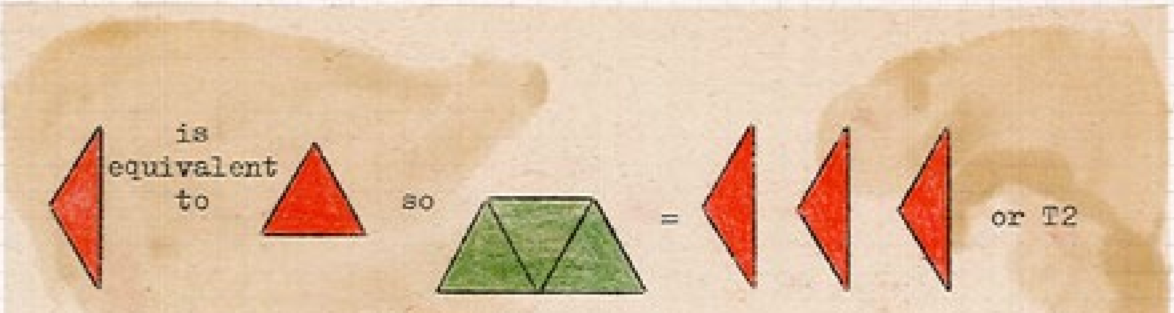


fig. 13.

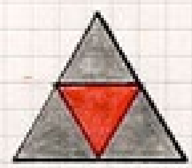
Presentation #4: The Ratio Between Figures: Between the Equilateral Triangle Inscribed in Another Equilateral Triangle.

Material: from T_1, the triangle T_1, the 4 red equilateral triangles (each 1/4 T_1)

1. Construct the equilateral triangle with the 4 fourths, each an equilateral triangle, following the black lines.

2. Superimpose this triangle on the grey triangle to verify congruency. They are equal.

3. Remove the three small equilateral triangles on the angles. Now the vertices of the red equilateral triangle coincide with the midpoints of the grey equilateral triangle. AND we know that the small red equilateral triangle is equal to 1/4 the large T_1. . .



SO

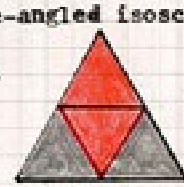
The inscribed equilateral triangle is equal to 1/4 the equilateral triangle which is circumscribed about it.

Presentation #5: The Ratio Between Figures: The Ratio of the Triangle Inscribed in the Hexagon: We have shown that it is equal to ½. That is, the triangle is equal to one-half the hexagon.

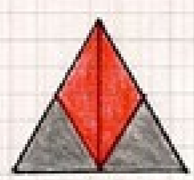
IV. V. Presentation #6: The Ratio Between Figures: The Equilateral Triangle Built on the Altitude of Another Equilateral Triangle.

Material: from T_1, the triangle T_1 + the two red equilateral triangles.
from H_2, the triangle T_2 + the two red obtuse-angled isosceles triangles.

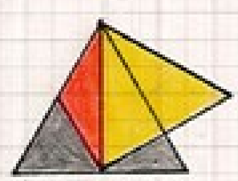
1. Construct the rhombus with two equilateral triangles.



2. Show that rhombus on T_1 so that the vertex of the rhombus coincides with the vertex and its opposite side's midpoint. Here the other opposite vertices of the rhombus coincide with the mid-points of the sides of T_1.



3. Construct the other rhombus with the two red obtuse-angled isosceles triangles. Substitute this rhombus in the triangle T_1.



4. Remove one-half: We have shown that the long side of this triangle, which is equal to the side of T_2 is equal to the altitude of T_1.
THE ALTITUDE OF T_1 IS EQUAL TO THE SIDE OF T_2.

4. CONCLUSION: We know that T_1 is to T_2 as 4 is to 3. When there exists a ratio between two equilateral triangles of 3:4, it means that the side of one corresponds to the altitude of the other. *Side of T_2 corresponds to altitude of T_1.*

Through this child-teacher collaboration in which both Dr. Montessori and I were involved some of these developments appeared really dramatic at the time, for search as we did they were no t to be found in any of the current geometry books. Here, for instance, is one.

IV.
When T1 is divided in four, one of the triangles resulting from it has one of its vertices in common with the apex of T1, the other with the midpoint of its base (see fig. 16). Together they form the rhombus which wee see to be congruent to theone formed by two obtuse-angled triangles of the small hexagonal box (ref. fig.12) V. If one places this second rhombus on T1 one sees that its long diagonal coincides with the side of T2 and as the latter's value is 3/4 of T1, one can make the following statement that "An equilateral triangle the base of which is equal to the height of another is 3/4 of the latter".

This aroused such enthusiasm in some children of the Amsterdamse Montessori School, then directed by Mrs. Joosten that by continuing they arrifed at the following conclusion. The ratio 3 to 4 found between T1 and T2 is also valid for H1 and H2 which are their double. So one can then also state VI that a hexagon incribed in another is 3/4 of the circumscribed one."

Thus working with the above boxes the following relationships between figures were found:
- An equilateral triangle inscribed in another is 1/4 of it;
- An equilateral triangle inscribed in a hexagon is 2/4 of it;
- An equilateral triangle built on the height of another is 3/4 of it;
- A hexagon inscribed in another is 3/4 of it.

The element used was always the equilateral triangle and its parts.

"All this is very interesting", most of the educator commented to Dr. Montessori, "but what is the use of it? Especially for such young children. Geometry is going to be eliminated from elementary schools, anyway". And eliminated it was, in many nations. I remember that at the time, at least in Italy, plane geometry was included mostly in the program of junior high schools. And it continued to be very dry.

A major difficulty was the theorem of Pythagoras. It was called "il ponte degli asini" (the bridge of donkeys) because many of the students failed to cross this bridge to higher studies. (Donkey was the epithet teachers regaled "stupid" pupils with, although even now I do not understand why. Because all who are familiar with donkeys know that they are quite intelligent animals.) When interrogated the pupils had to illustrate the demonstration by drawing on the blackboard. Those who were able to do it had had to memorize both the theorem and the demostration. But very few really understood. By drawing lines they had to produce parallelograms equivalent to squares, and rectangles equivalent to parallelograms and give the reason for the equivalence. The lines on the blackboard multiplied so that in the end the whole became a labyrinth which mirrored the labyrinths in the minds of the students.

Although the postulate that "the square built on the hypothenuse is equal to the sum of the squares built on the two sides" stuck in my mind, I must confess that the first time I really understood it was when I saw the demonstrations of Dr. Montessori given with appropriate metal insets to the eight-year olds. But in my mind, in the mind of Dr. Montessori and all those whom I interrogated, Pyt agoras' famous theorem stuck, and still sticks, solely related to squares. No other geometrical figure ever preten-

VI. Presentation #7: Ratio Between Figures: The Ratio of the Two Hexagons: We have proved that the ratio of H_2 inscribed in H_1 is 3:4.

Presentation #8: Ratio Between Figures: The Inscribed Square

Material

1. From the metal insets, the whole square, and the square divided into 4 parts by two diagonals.

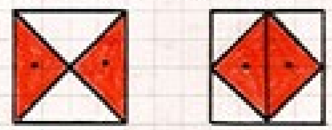
Presentation

1. Introduce the two squares, reviewing the construction of the fourths.

   Show that the four fourths are equal by superimposing.

1. This is the square. If we divide it by tracing two diagonals, we have four right-angled isosceles triangles. We can prove that all four triangles are equal.

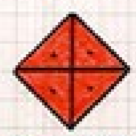
2. Remove two of the triangles symetrically, the two opposites. (Take



3. Then take one of the two fourths that are left out and slide the other one across. Then replace the second fourth opposite it to show the inscribed square.

3. Now the square is inscribed in the square of the frame. Because this square which is inscribed is formed of 2 of the four fourths, we have a square one-half the size of the square circumscribed about it.

4. GIVE PROOF: Show four eighths from the square inset of eighths (divided by diagonals) in the four corner spaces that are empty with the square inscribed. Then take those four eighths and form a square. We have shown that together these four triangles which completed the frame give half the square. We know that 4/8 is equal to 1/2. And we see that together they equal the same square which we have inscribed in the larger square. So THE SQUARE INSCRIBED IN ANOTHER SQUARE IS EQUAL TO $\frac{1}{2}$ THE LARGER SQUARE.



THE PYTHAGOREAN THEOREM

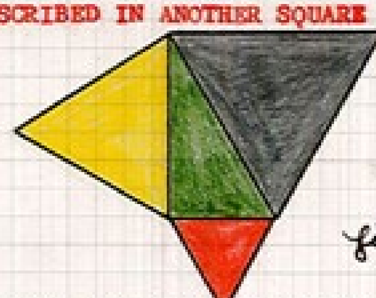


fig. 17

Materials

1. The three boxes of the constructive triangles.
2. An envelope????

1. Take from T_1 the two green halves of T_1. Classify them.

1. Each of these triangles is a right-angled scalene triangle. It is a special right-angled scalene triangle because it is $\frac{1}{2}$ of the equilateral triangle.

THIS MEANS THAT THE HYPOTENUSE IS EQUAL TO TWICE THE MINOR LEG. . .THE MINOR LEG IS EQUAL TO $\frac{1}{2}$ THE HYPOTENUSE.

2. Taking T_1, T_2 and the red equilateral triangle, demonstrate the corresponding weight, showing that the weight of T_1 (in one hand) would be equal to the weight of T_2 plus the red triangle. (shown in the other hand, the red on the yellow.)

2. We know that the red equilaterd triangle is equal to 1/4 T_1. AND that it is 1/3 of T_2. AND we know that T_2 is 3/4 of T_1. If I weighed them, T_1 would weight more. But if I add the red triangle to T_2, then they weight the same: $T_1 = T_2 + T_3$.

(T_3 is now the red equilateral triangle.)

3. We have said that $T_2 + T_3 = T_1$. An equilateral triangle built on the hypotenuse of the right-angled triangle is equal to the sum of the equilateral triangles built on the catheti.

ded to enjoy the same priveleges.

Let us consider again T1 and its divisions. Itis divided in two by the height ; Ifwe consider oneof the halves we see that it is a right-angled triangle the sides of which are: base =half of that of T1; height =the height of T1 and hypothenuse is equal to the side of T1.

Let us consider the sides now of the smaller equilaterals produced when T1 is didided in four. The sides of each of these four parts are equal to half the side of T1 and to the base of theright-angled triangle. Let us join T1 to the hypothenuse of the latter, T2 to the height and one of the four small triangles -Which weshall call T3 - to thebase of the latter. Taking T3 as unit of measure we find that the value of this is 1, the value of T2 andthe value of T1 is 4 (see fig;17).

Then we can state that the equilateral triangle built on the hypothenuse is equal to the sum of the equilateral triangles built on the other two sides. This equality 3+1 = 4 persists also when we multiply each of the terms by 2 we have 6+2=8; the resulting figures on the sides of the right-angled triangle will be rhombuses (see fig. 17a) . If we multiply each . we have a trapezium on each side of the right-angled triangle and similarly, if we multiply by 6, we shall have hexagons. Finally if we take the sides of the triangle as diameters, we shall have circles.(See fig. 18).

This is an illustration of what happens when the child is liberated from the slavery of textbooks and his mental potentialities are stimulated and helped by keys which enable him to investigate andto create in joy and enthusiasm. The contents of the school program are covered as well but alsomuch more and that with full understanding and with the joy of a seeking mind which is led to see relations, to reason, to become clear. Thishappens not only with regard to geometry but with regard to also the rest of mathematics. Isn't this also the aim of the advocates of "New Maths."? Perhaps, now the full essence of Dr. Montessori's work and concl clusions will be finally understood and appreciated.

Mario Montessori
"Communications" No. 1
1969. pp.12-18

(from: ASSOCIATION MONTESSORI INTERNATIONALE - COMMUNICATIONS.
Amsterdam - Holland - Koninginneweg 161. 1 - 1969. pp. 12-22).

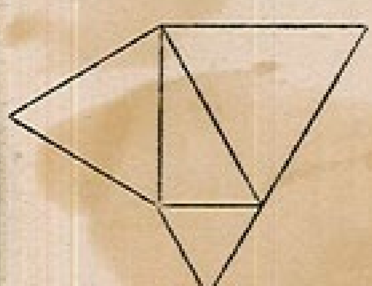


fig. 17.

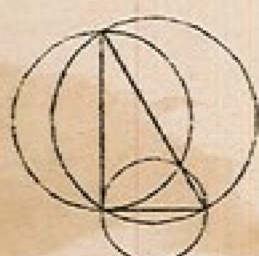


fig. 18.

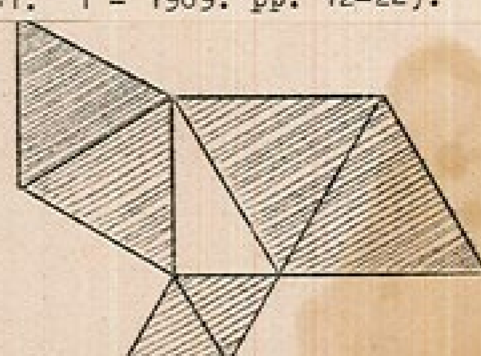


fig. 17 a.

Presentation #2: The Pythagorean Theorem Applied to Other Figures

Material:

1. The three boxes of the second constructive triangles.
2. An envelope which contains 3 additional figures equal to T_2, in cardboard.

Note: We have proved that $T_2 + T_3 = T_1$. Now we discover that this equality persists when we multiply each of the terms by the same number.

At 8 years this discovery is made on a sensorial level: the child takes from the three boxes those pieces which will form figures which first double each of the equilateral triangles built on the three sides of the right-angled triangle. (fig. 1). Then he adds a third equal figure (fig. 2), a fourth and fifth (fig. 3) and a sixth (fig. 4) to each of those sides. In each experience, he identifies the figure formed as a result of the new addition:

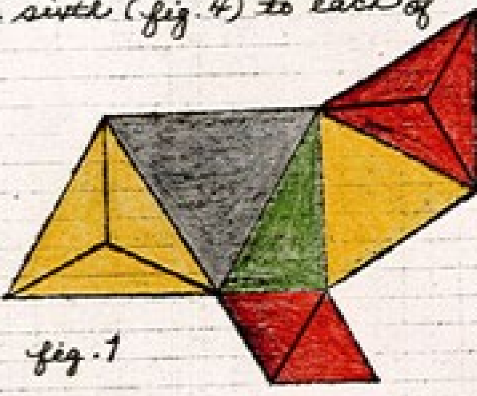


fig. 1

Rhombus

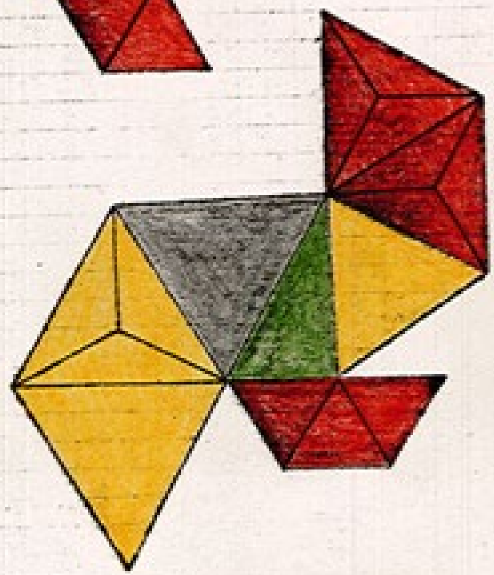


fig. 2

Isosceles Trapezoid

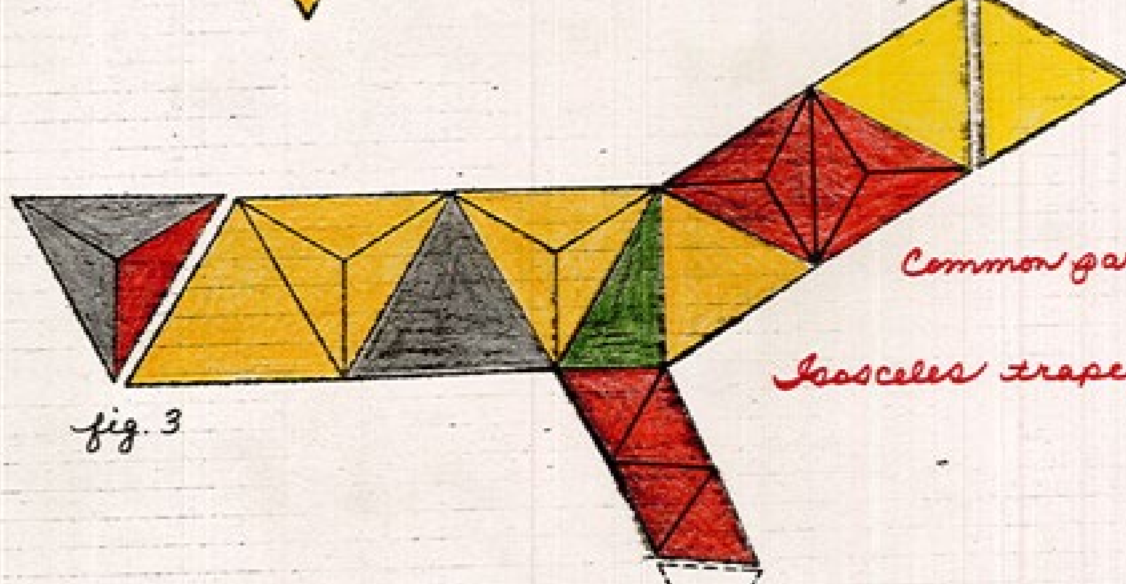


fig. 3

Common parallelogram

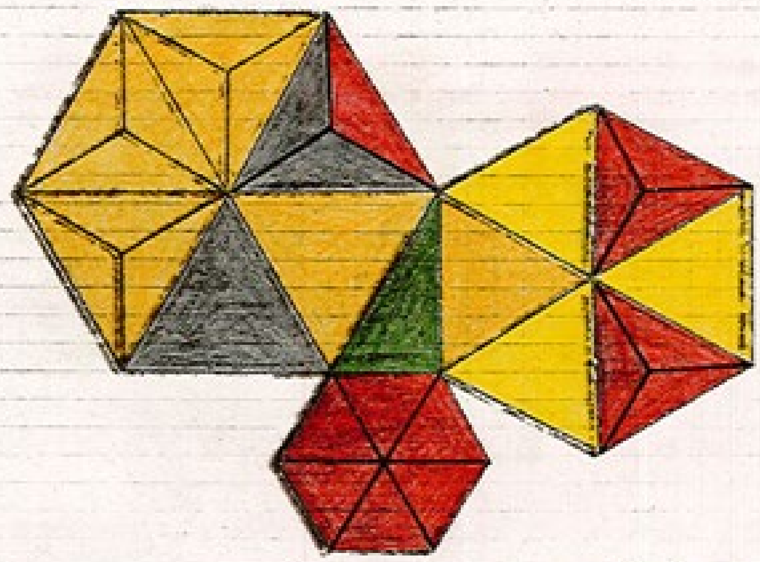
Isosceles trapezoid

In fig. 3, the additional cardboard figures are introduced. Shown in brighter yellow.

The Pythagorean Theorem applied to other figures...

With the passage to the hexagon in which the child has now taken the original equilateral triangle built on the sides of the right-angled triangle six times, we discover that:

the hexagon is equal to 2 × the isosceles trapezoid
and
the hexagon is equal to 3 × the rhombus.



Hexagon

fig. 4

At 10 years we can organize with the child the following table of calculations to represent this exploration.

n	T_2 + T_3 = T_1 $\frac{3}{4}$ + $\frac{1}{4}$ = $\frac{4}{4}$ 3 + 1 = 4	Resulting Figure
2	(2·3) + (2·1) = (2·4) 6 + 2 = 8	Rhombus
3	(3·3) + (3·1) = (3·4) 9 + 3 = 12	Isosceles Trapezoid
4	(4·3) + (4·1) = (4·4) 12 + 4 = 16	Common parallelogram
5	(5·3) + (5·1) = (5·4) 15 + 5 = 20	Isosceles Trapezoid
6	(6·3) + (6·1) = (6·4) 18 + 6 = 24	Hexagon

THE PYTHAGOREAN THEOREM applied to other figures. . .

The application of the Pythagorean Theorem can be extended to an amazing variety of figures. Using the right-angled scalene triangle from T_1 as the center on which the theorem figures are built, the child can show:

A. Regular polygons. Here the child constructs the regular polygons of cardboard, his guide the length of each of the three sides of the triangle. He can prove the equality of the two small figures to the large one by weighing them. LATER he finds the area of such figures and proves the theorem in that way. (slide example: regular octogons)

B. Semi-circles. Fractional sector of a circle: one-fourth. Circles inscribed in squares. (In this case, it is interesting that not only are the circles in the proper ratio, but also we may study that part of each of the squares not covered by the circle and find them to be another demonstration of the ratio.

C. Irreular polygons. (slide examples: rectangles, common quadrilaterals, acute-angled isosceles triangles.)

NOTE: In the work with the constructive triangles, the ratio prevails because each figure is, in reality, a compound figure, built of x units which have already proven the theorem as a unit. The secret uniting all the figures which demonstrate the theorem of Pythagorus is that the figures must be similar. So for the acute-angled isosceles triangles, if the angles are equal, since one side is established on each of the three sides of the right-angled triangle giving the side ratio; the three will be similar in a 1 - 3 - 4 relationship. So we can rely on either THE PROPORTION OF THE SIDES OR THE CONGRUENCY OF THE ANGLES.

D. The Curvalinear Triangle/ Triangle of Reuleaux Inscribed in a Square. The Triangle of Reuleaux minus that arc beyond the cord of one side which lies adjacent to the side of the right-angled triangle. The Inverted Curvaliner.

The figures noted above refer specifically to a series of slides shown by Sig. Grazzini as a preview of the publication of Pscho-Geometry. It will be a fine day when we see that volume come into print.

Dott.sa Montessori's comments on this interesting work with the theorem are particularly illuminating. She notes that proportional relationships exist first in life; and then in geometry. And so the small doll is made with a small nose, the middle-sized doll has a middle-sized nose and the biggest doll has the biggest nose. Then how shall we be surprised when that natural arrangement occurs in our geometrical consideration of the world of form?

Other materials introduced in the slides of today refer to geometrical designs which could be introduced as a possibility for the children's work if they don't think of it first. Particularly interesting examples:

A. Alternation of concentric circles and inscribed hexagons: experiment with color and with construction from the inside out and the outside in.
B. The Babylonian figure: the triangle inscribed in the circle inscribed in the square inscribed in the hexagon. (Perhaps in another order)
C. Regular polygons inscribed concentrically, as seen in the infinity work.
D. Graduated polygons with a common angle.
E. Graduated ellipse or oval with one point in common.
F. Circles alterantely tangent. (Also possible with the ellipse and oval.)