DECIMAL NUMBERS

The work with fractions must necessarily preceed this chapter. The child must have understood well the consideration of the whole divided into parts. The study of decimal numbers has two chapters: 1) numeration and 2) operations.

I. Numeration

Before introducing decimal numbers, we review the value of the whole numbers with the child, emphasizing their relationship to the unit. The important point is that the value of the unit is the basic element of every number, whole or decimal. And its value is, thusly, the center of our study in the decimal numbers. In order to make the unit more interesting, we may compare it to the tree's trunk which supports all the other parts of the tree. Or the candellabra which has many arms, but the middle supporting part is the unit arm.

The decimal numbers are formed in two ways: 3.436575 (whole number + decimal number)
.048 (only the decimal number)

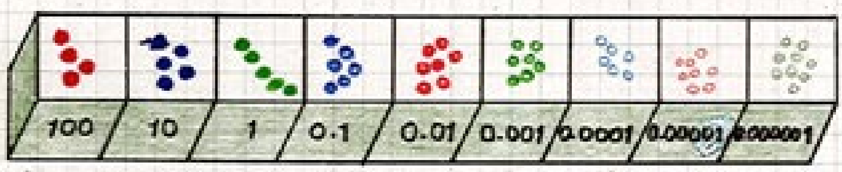
The first three digits after the decimal point form the class of the thousanths(tens, hundreds and thousand thousandths) and the next three, moving left to right, form t he class of the millionths (ten-thousandths, hundred-thousandths, millionths). That is, the name of the class is given by the last digit in each group of three.

In reading the decimal number as shown in case #2 (0.048) we say "Zero units and thirty-eight thousandths" OR simply "forty-eight thousandths."

Presentation #1: The Quantities

Material

1. From the circle metal insets, the whole and the circle divided intotenths.
2. A cardboard section of the circle (red), the value of 1/10th.
3. Scissors
4. One green unit
5. A long rectangular box containing the whole and decimal materials: there are nine divisions in the box; moving from left to right, a section for the red hundred beads, the blue ten beads, the green unit beads. Then six sections which contain materials representing the decimal numbers, tiny flattened pill-shaped discs; the first division containing dark blue pills for the tenths, the next dark red pills for the hundredths, then dark green pills for the thousandths. The next series of three are in lighter colors of the hierarchical hues to implicate the decreasing value of the pills. So we have lighter blue pills for the ten-thousandths lighter red pills for the hundred-thousandths, lighter green for the millionths. The box is light green; and on the front are labeled the numerical contents of each section.



Presentation

1. Show the whole circle inset and then the green bead. Note the equality of value.

1. This circle represents the whole. This bead has the value of 1. . .unit It is one unit because there is no other value indicated for it by position. So the circle has the same value as the bead---ONE WHOLE UNIT.

2. Show the whole divided into tenths.

2. Here we have a figure which is congruent to our whole circle; and is in fact still the whole circle. But it is divided into 10 parts. What is the value of each section of this circle? One-tenth.

THE DECIMAL NUMBERS. . .
Transformation of an Ordinary Fraction to a Decimal Fraction to a Decimal Number. . .

NOTE: The child may do this work with all the fractions of the circle. He uses the centesimal circle as an instrument to translate the fractions to decimal fractions to decimal numbers without calculation. That is, this work is strictly at a sensorial level. The work is a preparation for later work and we return to it for proofs.

The child can also do addition with the centesimal circle and the fractional insets without changing denominators: 1/4 + 1/5 = 45/100, using the centesimal circle as he did the Montessori protractor. He can also make addition proofs at this sensorial level when working with fractions.

Presentation #4: A Second Preparation for Multiplication: Reviewing Multiplication with the hierarchies of 10, 100, 1,000.

Material

1. The numeral cards for the decimal system from the bank game.
2. The series of the decimal number symbols.
3. From the bank game, the grey zeros (one, two, and three) cards, and the three series of grey numeral cards from 1 - 9 used in the bank game for the multiplier.
4. The box of beads and discs for decimal numbers.
5. The yellow board for decimal numbers.

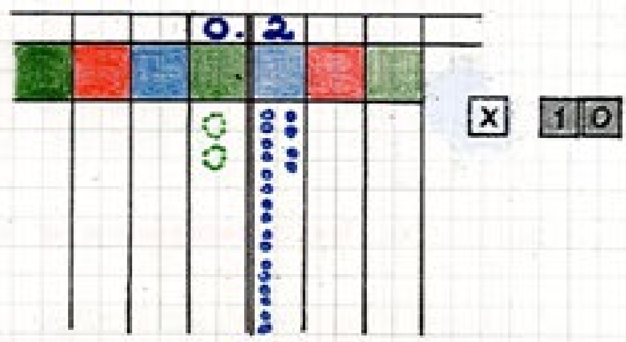
Presentation

1. Present the prepared problem: 0.2 X 10 =

2. The child shows the problem on the mat with the symbol cards.

0.2 x 10 =

0.2 x 10 =



3. Then he substitutes the cards 1 and 0 from the grey series for the blue ten which he puts aside.

4. He now shows the multiplicand at the top of the board, the operation signs and the multiplier to the right side.

5. Then he must take 0.2 ten times with the quantities.

6. He counts his result: changes the 20 tenths for 2 units.

0.2 x 10 = 2

7. He replaces the symbol cards together on the mat, returning the original numeral symbol for the multiplier and shows the answer.

8. He copies the operation in his notebook.

9. Second operation: 0.02 X 10 = 0.2
This is an important link to the next two operations.

9. Here I must take two hundredths 10 times. I discover that it is equal to two tenths.

10. Third operation: 0.02 X 100 = 2.
Here the multiplier 100 is substituted by the grey card 1 and 00.
He uses the product discovered in the last operation and then multiplies by 10.

10. If we take 0.02 one hundred times, our multiplication will take a very long time.
So let's take 0.02 ten times.
WE KNOW THAT GIVES A PRODUCT OF 0.2.
But we've only multiplied by 10---
we must multiply by 100. What can we do?
We multiply that product by 10.
0.2 X 10 = 2.
We know that when we multiply by ten we simply change to the next higher hierarchy. . .SO OUR ANSWER IS 2.

11. Fourth Operation: 0.002 X 1,000 = 2
Again the child, rather than multiplying by 1,000, begins the process by multiplying the 0.002 by 10. He should know at this point that multiplication by 10 means simply that he moves to the next hierarchy. (If he doesn't know for sure, he can again show this multiplication with the quantities on the board) After multiplying by 10, he must again muliply by 10 and finally a third time, since he knows that his multiplier 1,000 is 10 X 10 X 10. So his first product is 0.02, his second product is 0.2 and his final product is 2.
NOTE: This particular process emphasizes the importance of the work with the powers of 10.

12. In each operation, the child has recorded the work in his notebook. Now we call his attention to the list he has made.

When we multiply by 10 we see that we have moved up one hierarchy.
When we multiplied by 100, the movement was two hierarchies to the left on the board.

NOTE THE MOVEMENT OF THE DECIMAL POINT:

And we can observe that when we move to a greater hierarchy, the decimal point moves one place to the right, increasing the value of the digit by one power of 10.

CONCLUDE WITH A RULE:

Therefore, I must move the decimal point as many places to the right as there are zeros in the multiplier.

And verify it with the child's list.

MULTIPLICATION WITH THE DECIMAL NUMBER BOARD: 3 CASES

CASE 1: Decimal Number X Whole Number
A. 0.25 X 4 =
B. 2.31 X 3 =

CASE 2: Whole Number X Decimal Number
A. 4 X 0.25 =
B. 3 X 2.31

CASE 3: Decimal Number X Decimal Number
A. 2.4 X
B.

Material
1. The decimal number board and the box of beads and discs.
2. The decimal number symbol cards.
3. The numeral cards of the decimal system, from the bank game.
4. The bank game grey series: 0, 00, 000 and three series of 1-9.

Presentation #5: Multiplication CASE 1A: Decimal Number X Whole Number

1. Present the operation: 0.25 X 4 = . The child shows the problem with the numeral cards, using the decimal number series and the decimal system series. 0.25 x 4 =

2. Then he substitutes the grey 4 (multiplier) for the numeral card 4; and he shows the multiplicand decomposed.

0.05 x 4 =
0.2

3. He takes first the symbol 0.05 (the longest card, the least value) and shows it on the board. He places the operations symbol and the grey 4 to the right and makes the multiplication with the quantities, taking five hundredths four times. EXCHANGES THE 20 HUNDREDTHS FOR TWO TENS.

4. Leaving the quantities on the board, he removes the first symbol card (0.05) and turns it over on the mat. Then he shows 0.2 on the board and does the multiplication times 4. HE NOW SHOWS ON THE BOARD TEN TENTHS. HE EXCHANGES FOR ONE UNIT.

5. He recomposes the multiplicand on the mat, brings back the original numeral card for the multiplier and shows the answer with a symbol. He copies the operation.

NOTE: SECOND LEVEL : Very quickly the child makes the passage of multiplication to greater abstraction in his operations on the board. He can do 0.05 X 4 by saying five hundredths times four equals twenty hundredths which is equal to two tenths and he shows the quantity two tenths. The higher level comes quickly because he discovers the process of exchanging is eliminated, the operation goes faster, and the result is the same.!!

Presentation #5: CASE 1B: 2.31 X 3 = 6.93

The process here is the same as in CASE 1A: but now his operation includes the multiplicand with both the whole number and the decimal number. In the cards which he takes to show his number, then he must compose the whole decimal number and then take the numeral card for the unit digit and show them together.

With the presentation of every case, we give several examples and then play a game:

GAME: Show an operation with the symbols, turning face down the multiplicand:

? X 4 = 1

What is the multiplicand?

We know it must be a decimal number because the answer is 1. And that product is a result of a multiplication by 4. So the multiplicand must be smaller than the unit.

How much smaller than the unit? How many times smaller?

It is a decimal fraction which, when multiplied by 4, gives 1.

Then I must look for 1/4 part of the unit.

The child must recall the work of transforming an ordinary fraction to a decimal fraction. If he doesn't, we reason that one-half of the unit is 50/100; there are two fourths in each half; therefore, each fourth is 25/100.

OR

The symbols which we turn down and try to discover may be the multiplier.

0.25 X ? = 1

It is important that we help the child relate this 0.25 to the ordinary fraction 1/4. Such exercises provide an important link to the work with the fractional work with the circle insets.

Presentation #6: CASE 2A: 4 X 0.25 : Whole Number Times a Decimal Number

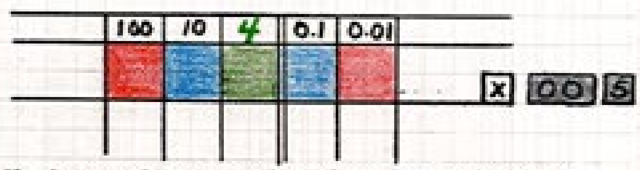
4 x 0.25 =

4 x 005 =
02

1. We present the operation and the child shows it with the symbol cards.

2. Then he substitutes the multiplier symbols with the grey digits and zeros; then makes the decomposition of the multiplier.

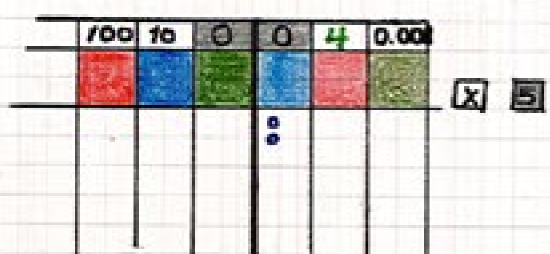
3. He shows the 4 digit on the board and the first part of the multiplier to the right.



4. Then he changes the multiplier to a whole number, thereby dividing the multiplicand by 100 and multiplying the multiplier by 100. He takes the two grey zeros from the multiplier and shows them placed to the left of the unit digit on the board, repositioning to show 0.04. He multiplies.

We know that multiplying 4 X 5/100 is the same as multiplying 4/100 X 5.
So we change the multiplier to a whole number.
In order to do this I have to multiply the 0.05 times 100---then I must divide the multiplicand 4 by 100.

5. The child then replaces the zeros back with the multiplier, turns it face down on the mat and shows the multiplier 0.2. Here he must multiply the 0.2 by ten and divide the 4 by ten---again he places the grey zero to the left of the whole digit and it becomes 0.4. He makes the multiplication with quantities.



6. To complete the operation he recomposes the multiplier, brings back the original symbol cards for the multiplier, and, reading the answer from the quantities on the board, he composes the product with the numeral cards.

NOTE: In order to do this work, the child must understand the hierarchies well. It requires multiplication and division by ten at the same time. But it is bringing him towards abstraction and greater mental flexibility.

7. The child does several more operations showing a whole number times a decimal number. . .THEN recording each operation in his notebook. Looking then at his record of the operations, note that WHEN THE MULTIPLICAND IS A WHOLE NUMBER AND THE MULTIPLIER IS A DECIMAL NUMBER (that is, a fraction of the whole), THE PRODUCT WILL ALWAYS BE SMALLER THAN THE MULTIPLICAND. We are multiplying 4 times 0.25---1/4 of the whole and our result is 1. It is smaller than the multiplicand. It is one-fourth of the multiplicand. AN IMPORTANT REFERENCE TO THE DECIMAL NUMBERS AS A FRACTION.

   NOTE: See also the presentation of a whole number by a fraction in the fraction work.

Presentation #7: Whole Number Times a Decimal Number which is formed of a whole and a decimal part.
CASE 2B: 3 X 2.31 =

1. The operation proceeds as in the previous presentations: the child forms the operation on the mat with the decimal system and decimal number numeral cards. He substitutes the grey numerals and zeros for the multiplier. (He decomposes that multiplier first in order to see the necessary substitutions; also a guide to the subsequent multiplication.) Then he shows on the board the multiplicand 3. And to the right of the board he shows his first multiplier: 0.01 formed of the double grey zero card and the grey 1. NOW IT IS NECESSARY TO TRANSFORM THE MULTIPLIER INTO A WHOLE NUMBER WHICH HE DOES BY PUTTING THE TWO ZEROS TO THE LEFT OF THE 3 ON THE BOARD, THUS TRANSFORMINGTHAT 3 INTO 0.03 WHICH MOVES THE NUMERAL 3 TWO PLACES TO THE RIGHT. (Here the child has simultaneously multiplied the multiplier by 100 and divided the multiplicand by 100----this is possible because to multiply 3 X 0.01 is the same as multiplying 1 X 0.03.) Now he shows the multiplication by this first multiplier with the bead quantities (and discs)---here three red discs (0.03). The product is three hundredths. He replaces the grey zeros with the 1 and removes that multiplier to the mat, turning it over. And repeats the process with the 0.3--- now multiplying the 0.3 by ten and dividing the multiplicand by 10.

2. Finally, the new element here is the multiplication of the last part of the multiplier (which is the whole number 2) times the multiplicand 3. The product will be 6 green beads on the board. HERE IT IS NOT NECESSARY TO TRANSFORM THE MULTIPLIER BECAUSE IT IS ALREADY A WHOLE NUMBER.

3. The child copies the operation and the result in his notebook, reading the product from the materials shown on the board.

Presentation #8: Decimal Number times Decimal Number: CASE 3A: 2.4 X 3.6 =

In this multiplication the child will perform FOUR operations: first he multiplies the two elements of the multiplicand (shown when decomposed) times the SMALLEST hierarchy of the multiplier: 0.6.

and	0.4 X 0.6 2 X 0.6	give the FIRST PARTIAL PRODUCT
Then. . and	0.4 X 3 2 X 3	give the SECOND PARTIAL PRODUCT

With the materials we follow these four passages; but the work proceeds as in the previous operations: showing the operation as numeral cards, decomposition of BOTH multiplicand and multiplier, substitution of the grey cards, trasnformation of the multiplier to a whole number when necessary (first partial product); recomposition of operation and showing the result with the numeral cards. HERE IN THE PRODUCT FORMATION we must form both the whole part and the decimal part with the corresponding cards--- and then combine them to show the whole product. The child copies the operation.

AN IMPORTANT CONCLUSION TO THIS PRESENTATION:

We call the child's attention to the hierarchies which he has multiplied and the resulting hierarchy: Multiplying tenths times tenths gives hundredths!! Just as when we multiplied tens times tens and our result was hundreds. (See checkerboard work---preparation for decimal number checkerboard.) Then hundredths times the unit gives hundredths---hundreds times the unit gives hundreds., . . .SO when we multiply tenths times tenths, we get a product which is 100 times less than the units and ten times less than the tenths.

Presentation #9: Decimal Number times Decimal Number without Whole Numbers:
CASE 3B: 0.74 X 0.26 =

If the child has understood the important work of the previous presentation, he will see immediately that the result of this problem will be at ten-thousandths!!! Hundredths times hundredths always gives ten-thousandths. (Parallel concept with that which the child is getting in his work with surface measures---passage to algebra---to square root--SEE CHART OF SQUARES)

Here again the operation is performed in four operations:

	0.04 X 0.06 =	
and	0.7 X 0.06 =	gives the FIRST PARTIAL PRODUCT
Then .	0.04 X 0.2 =	
and	0.7 X 0.2 =	gives the SECOND PARTIAL PRODUCT

The work proceeds as in the previous operations. The emphasis is on the hierarchies which are multiplied and the resulting hierarchy.

0.04 X 0.06 = A multiplication of hundredths by hundredths will give ten-thousands. We accomplish this by transforming the 0.06 to a whole number---multiplying the 0.06 times hundred and dividing the 0.04 by hundred. Thus the 0.04 becomes 0.0004, the numeral is moved two places (two hierarchies to the right on the board). Then the multiplication is ten-thousandths times units which gives ten-thousandths. That is,

0.0004 X 6 gives 0.0024 OR two thousandths and four ten-thousandths

Last product: 0.7 X 0.2 = A multiplication of tenths times tenths will give hundredths. Here again we transform the 0.2 to a whole number. Then our multiplication is hundredths times units. The result is 0.14 OR one tenth and four hundredths.

It is important to analyze each part of the multiplication in this way at this point in the multiplication work. When the answer has been formed on the board then, we can conclude that our previous statement has been confirmed: hundredths times hundredths have given us a product which arrives at ten-thousandths: 0.1924.

NOTE: Montessori notes that here we are transforming the multiplier into a whole number. With other material we can also transform the divisor into a whole number in the same way.

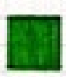
THE CHECKERBOARD FOR THE MULTIPLICATION OF DECIMAL NUMBERS

Presentation #1: Forming the Checkerboard for Decimal Numbers

Material

1. Cardboard squares (49 total) which are 10 cm^2 in the following colors: 12 dark green, 9 dark blue, 7 dark red; 9 light blue, 7 pink, 5 light green.
2. Four number strips (shown in figure).
3. The decimal number checkerboard (also shown in figure.)

1. Present first the dark green unit square.

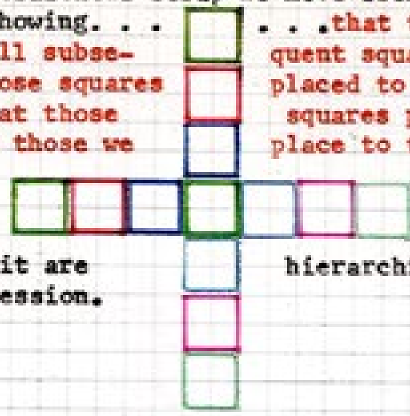


1. This square is the center of our work. It represents the unit.

2. Show first the center horizontal strip, developing it square by square alternately to the left and right (whole and decimal number) as the child has done both with the decimal materials and the numerals (in the exercises of the crowning of the unit. Analyze each square according to position: what do we have to the left of the unit? tens. To the right? Tenths. . . . . . .

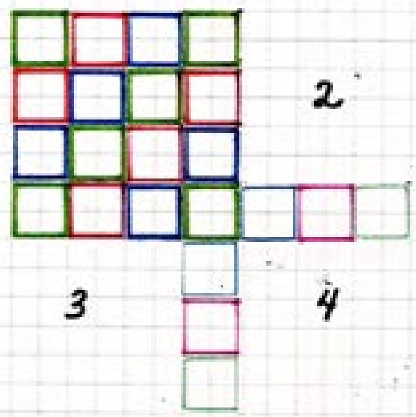
Our horizontal formation extends, then, to millions to the left and to millionths to the right.

3. Now to that original horizontal strip we move from the unit up three squares and down three squares. Showing. . . . . .that the square above the unit increases by one hierarchy and all subsequent square moving up increase by one hierarchy. . . .just as those squares placed to the left of the unit increase by one hierarchy. And that those squares placed below the unit DECREASE by one hierarchy, just as those we place to the right of the unit. THUS WE PRESENT THE RULE OF FORMATION : squares to the left and above the unit are hierarchies larger than the unit in the natural progression: and squares to the right and below the unit are hierarchies smaller than the unit in the natural order of progression.



4. Remove the squares, and begin again, starting with the unit, then asking the child to form the previous cross formation.

4. Now we are going to do the same work in another way.
We are going to form a big square.



5. Begin by filling in the quadrant top left which will all be whole number squares: the child does the work: we help with the analysis of the formation:
We place a square now one place to the left of the unit and one hierarchy above. What will it be? It is greater than the unit---a whole number. And it is one hierarchy above the tenths---we need hundreds. CONTINUE, FORMING THE ENTIRE FIRST QUADRANT. Movement for the construction should be horizontally across each successive row from right to left; then up a row.

6. The child constructs the second quadrant: here we have hierarchies which decrease as we move to the right; but which increase as we move up. Thus we have to the right of the unit the tenth square on the center horizontal row; BUT TO THE RIGHT OF THE TEN SQUARE WHICH IS DIRECTLY ABOVE THE UNIT CENTER, WE HAVE ONE HIERARCHY LESS WHICH IS THE UNIT.
<u>Third quadrant</u>: here the square is placed which is one hierarchy greater than that square shown on the center vertical strip---that is one hierarchy greater than the decimal part shown because the movement is to the left BUT IT IS ALSO A HIERARCHY LESS THAN THAT ONE ABOVE. We discover then that our first square is the unit: one hierarchy greater than the tenth and one smaller than the ten.
<u>Fourth quadrant</u>: Now we have the hierarchy which is smaller than that one above and also smaller than that one to the left---the movement is down and to the right. Therefore, our first placement is the square of the thousandths, which is one hierarchy less than both the tenth above and to the left.
THUS THE CHILD FORMS THE WHOLE CHECKERBOARD.

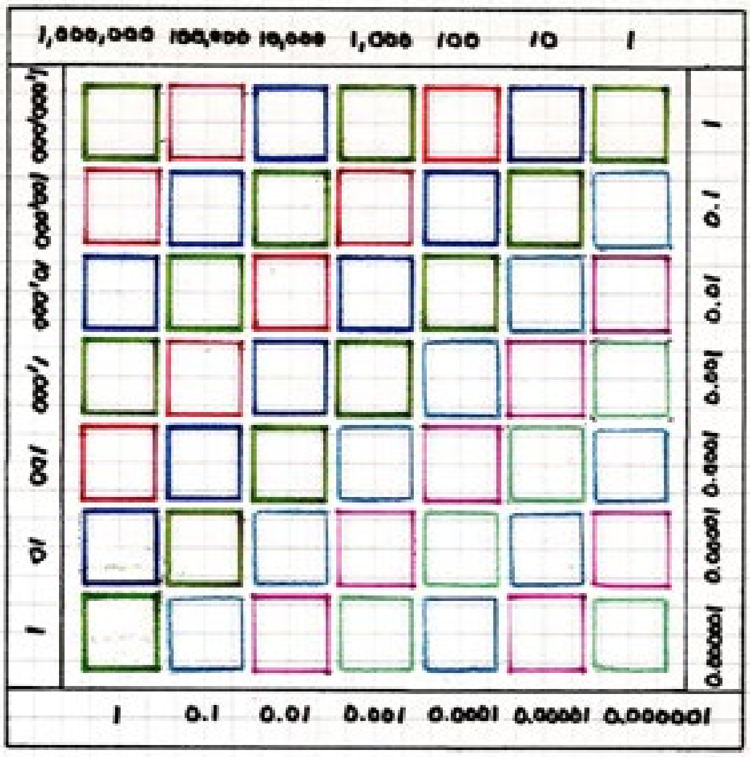
7. Present the decimal number checkerboard. Show that it is the one already constructed. It still has no identifying number strips.

7. This checkerboard corresponds to the one that we have just formed.
We must define the value of each square because there are no numbers on this board.

8. Define the position of the board. SHOW THE BOARD IN OTHER POSITION S, NOTING WHY EACH OF THE OTHER THREE BREAK THE RULES OF FORMATION.

There is only one valid position for the board according to our rules of formation.

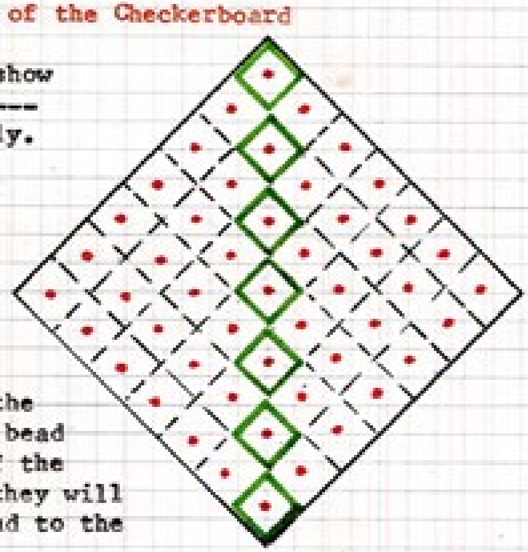
8. How shall we orient the board?
When we formed our square first, we had the unit at the center, the larger hierarchies to the left and to the top---the smaller ones to the right and down. SO. . .that formation defines our position. And the colors indicate which are our greater and smaller hierarchies.



9. Introduce the four number strips: we note that the numbers correspond to the value of the squares. Thumb tack each of the four strips in place as shown in figure.

NOTE: The board we are describing is handmade, without the numbers actually printed on the sides of the board. If one uses the manufactured board, we have those numbers shown around the square. For this presentation, then, use paper strips over the numbers until the end of the presentation when we reveal the number strips.

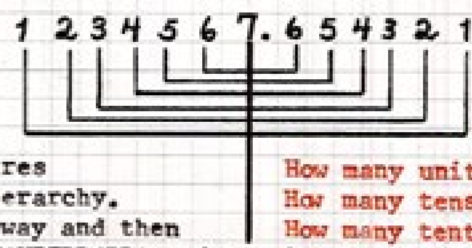
Presentation #2: Finding the Decimal Value of the Checkerboard



1. Position the board on the diagonal and show a unit bead on each of the unit squares--- the center diagonal, now shown vertically.

2. Then place a unit bead by alternating hierarchies (tens, tenths, hundreds, hundredths, etc.) on each square of the checkerboard.

NOTE: If we show the board in this way and place the same bead bar on all the squares, we discover that we have the same value on the diagonal now shown vertically; that is, our bead bars will have the value of units on each of the center diagonal squares. THEN we see that they will have different values moving to the right and to the left according to hierarchy. This again recalls the crowning of the unit and the arrangement of the DECIMAL NUMBER BOARD---GREATER VALUES TO THE LEFT AND SMALLER TO THE RIGHT.



How many units? 7
How many tens? 6
How many tenths? 6 . . .

3. With the child analyze the number of squares there are on the checkerboard for each hierarchy. Consider each of the hierarchies in this way and then NOTE THAT THIS ORGANIZATION RECALLS THE CANDELLABRA. (above)

4. Read the number given by this analysis: the number then summarizes the value of the checkerboard. We have also shown how many squares of each hierarchy form the board.

DECIMAL NUMBERS. . .
The Checkerboard. . .

Presentation #3: Familiarization with the Board

1. Place a bead bar on the unit square at the center: Then move that same bar to other squares and the child identifies the changing value.	1. What is the value of this bar here? 3 What is its value here? 0.03 Here? 0.3
2. Using two bead bars, form a quantity on the center unit square and the ten square immediately to the left. The child reads the value. Then show that the same value is given on those two corresponding squares all along the diagonal.	2. What is the value of this quantity? 25 Is this the only way we can show this value? NO We can show 25 in 6 positions on the diagonal.

NOTE: During this familiarization WE DO NOT SHOW A QUANTITY FORMED VERTICALLY IN PART OR WHOLE: because we are preparing for the work in which the multiplicand is always shown horizontally on the board.
However, it is important that he see that a quantity can be formed in several positions on the diagonals---and in each case the quantity has the same value.

3. Say a number: 5.23 and ask the child to form the quantity on the board with the bead bars. Increase the difficulty, including ZEROS and MORE DIGITS.

Continue the exercises of forming quantities until he understands the board well. THEN proceed to multiplication . . .

Presentation #4: Multiplication: Decimal Number X Whole Number   3.089 X 2,726 =

Material
1. The decimal number board.
2. The box of bead bars.
3. The numeral cards for the whole numbers from the decimal system materials and the three series of decimal number cards.
4. Three series of white numeral cards with black numerals (for multiplicand.) 1 - 9
5. Three series of grey numeral cards with black numerals, each containing the numerals 1 - 9 (for the multiplier.)
6. To the series described in #4 and #5, TEN WHITE ZERO CARDS AND TEN GREY ZERO CARDS.

NOTE: On the checkerboard, the multiplicand is always formed on the horizontal rows.
The multiplier is always shown on the number strip to the far right side of the board, the unit coinciding with the horizontal row on which the multiplicand is placed.
The product will be read on the bottom row and the lower left column.

1. The child forms the operation on the mat with the corresponding numeral cards from the decimal system materials and from the decimal number series with the operation symbols.

2. Then he substitutes the LOOSE WHITE NUMERAL CARDS FOR THE MULTIPLICAND and the LOOSE GREY NUMERAL CARDS FOR THE MULTIPLIER, showing them above the operation on the mat. That is, we leave the original operation on the mat during the entire operation, but we actually do the work with the white and grey cards, so we begin by showing that we are using the same numbers as shown.

3. The multiplicand is first placed on the board: white numeral cards. BECAUSE WE HAVE THREE DIGITS FOLLOWING THE UNIT IN THE MULTIPLICAND, we must show the units (3) on the center unit square. That is, it is important, particularly in the first work done in multiplication with this board, that the last cipher of the multiplicand coincide with the last square on the horizontal row at the right. THIS IS A KEY TO THE PLACEMENT OF THE MULTIPLIER DIGITS.

4. The multiplier is then shown on the number strip at the right side of the board, THE GREY UNIT DIGIT COINCIDING WITH THE HORIZONTAL ROW ON WHICH THE MULTIPLICAND IS SHOWN***with the last digit of the multiplicand. In this particular operation, then, the other digits of the multiplier are shown above the unit digit, because they represent whole hierarchies---greater than the unit.

5. FIRST LEVEL WORK: The child does the multiplication as in the first level of the whole number checkerboard work, showing each product as the number of bead bars which represent the "times taken" indicated by each successive digit of the multiplier. He begins by multiplying all the hierarchies of the multiplicand times the units of the multiplier (6). Then he turns that grey numeral card face down on the board and begins the multiplication of each digit of the multiplicand times the tens of the multiplier (2). In each multiplication, then, the result is the multiplie number of bead bars. So .009 taken 6 times is shown as six nine-bars ON THE LIGHT GREEN THOUSANDTHS SQUARE. . .
THE IMPORTANT PART OF THIS WORK is the verbalization of each multiplication.
Each time the multiplication is made, we verbalize the units of the multiplication, thus reviewing the hierarchies which, when multiplied, give another hierarchy.
THE MULTIPLICATION IS NOT THE PROBLEM NOW, BUT RATHER THE IDENTIFICATION OF THAT HIERARCHY WHICH IS THE RESULT OF THE MULTIPLICATION OF TWO HIERARCHIES. In our first case, then, we are always multiplying one whole hierarchy times one decimal hierarchy.
What hierarchy do we obtain when we multiply hundredths times units? Hundredths.
THE CHILD WRITES THIS OUT WITH EACH MULTIPLICATION:

0.009 X 6	= 0.054	Thousandths X units = thousandths
0.08 X 6	= 0.48	Hundredths X units = hundredths

In each case, the control is the color of the square which gives the hierarchy together with the number strips which identify each diagonal.

6. Having completed the multiplications times each digit of the multiplier, the child forms the partial products on each horizontal row, reducing the bead bars shown to only ONE PER SQUARE, as in the reduction of the whole number checkerboard. SO six nine-bars are reducing to one four-bar on the thousandths square and a five-bar on the hundredths square to the left, where it is combined with those bars shown already on the hundredths square in the next reduction.

7. Slide all the quantities down the diagonals to the bottom row and far left column, depending on where the diagonal of the hierarchy ends.

8. Here a reduction is again made to show only one bead bar per square. And the result is read on the board. THEN THE RESULT IS SHOWN WITH THE NUMERAL CARDS ON THE MAT WITH THE ORIGINAL OPERATION. The child copies it in his notebook.

NOTE: The work at this first level is short because it involves the longest operation, and the child quickly passes to the second level. It is an important level, however, because we return to it for the design work, just as we did with the whole number checkerboard.

Presentation #5: SECOND LEVEL WORK: Decimal Number times Whole Number 3.089 X 2,726 =

1. The work proceeds as described in presentation #4, steps 1, 2, 3, and 4.

2. The work of multiplication is now identified as a partial product with the completion of each horizontal row of products shown. And this multiplication is now shown as a combination of bead bars which represent the product: that is, 0.009 X 6 is shown as the four-bar on the thousandths square and the five-bar on the hundredths square to the left. SO. . .the child multiplies all the digits of the multiplicand by 6 in this way; then reduces the products to one bead bar per square and READS THE PARTIAL PRODUCT: HE MAY ALSO WRITE NOW THE PARTIAL PRODUCTS AND HE DOES THIS WITHOUT THE DECIMAL POINT. In reading the partial product, again the colors of the squares and the numbers on the strips indicate the hierarchies, serving as a guide.
NOTE: the child may quickly pass to the process of changing each product to one bead bar per square as he makes the multiplications, thus eliminating the step of reducing to the partial product at the end of each horizontal row.

3. VERBALIZATION continues to be important. Each product is identified as a hierarchy as it is placed. So eight hundredths taken seven hundred times gives fifty-six units. THE CHILD IS WORKING WITH THE BOARD TO LEARN HOW THE HIERARCHIES ARE FORMED: BY WHAT HIERARCHICAL FACTORS. He continues to write this analysis.

4. When each partial product has been formed, the hierarchies are brought down on the diagonal, the final reduction made to one bar per square, and the answer read.

Decimal Numbers...
Multiplication with Checkerboard : Second Level...

3.089 × 2,726 =

First Partial Product

.009 × 6	= .054	thousandths × units = thousandths	
.08 × 6	= .48	hundredths × units = hundredths	
.0 × 6	= 0		
3. × 6	= 18.	units × units = units	
	18.534		

Second P.P.

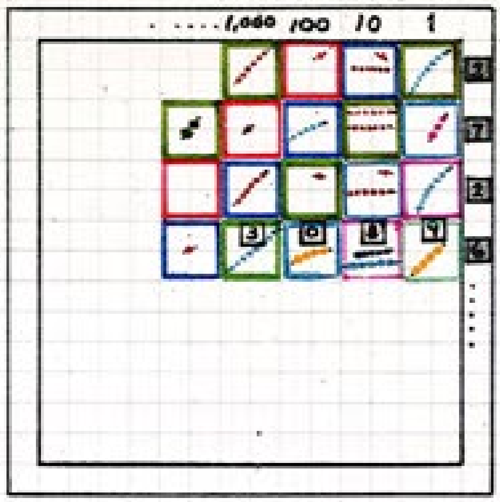
.009 × 20 = .18	thousandths × tens = hundredths	
.08 × 20 = 1.6	hundredths × tens = tenths	
.0 × 20 = 0		
3. × 20 = 60.	units × ~~units~~ tens = ~~units~~ tens	
61.78		

Third P.P.

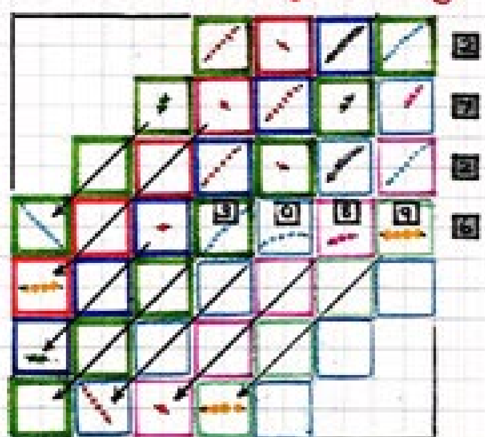
.009 × 700 = 6.3	thousandths × hundreds = tenths	
.08 × 700 = 56.	hundredths × hundreds = units	
.0 × 700 = 0		
3. × 700 = 2100.	units × hundreds = hundreds	
2162.3		

Fourth P.P.

.009 × 2,000 = 18.	thousandths × thousands = units	
.08 × 2,000 = 160.	hundredths × thousands = tens	
.0 × 2,000 = 0.		
3. × 2,000 = 6,000.	units × thousands = thousands	
6,178.		



#2 Reducing each horizontal row to the partial product...



... brings each hierarchy together down the diagonal to the corresponding square.

#3 The important work is the writing of the hierarchical factors, which give the hierarchy corresponding to the product square.

#4. and reduced to show the final product which he reads from the board, then calculates his addition of the partial products and checks the two results.

The child writes the partial product by reading it from the materials on the board. NOT AS THE CALCULATION ABOVE.

```
    3.089
  × 2.726
  -------
   18 534
   61 78
 2162 3
 6178
 --------
 8420.614
```

Partial products written without decimal point. The position of the decimal point is determined by the quantity as shown on the board in the final product... BUT

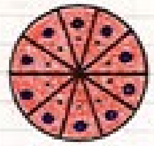
as the child becomes increasingly aware of the hierarchical products, he sees by looking at the two factors of the multiplication what the units of the answer will be.

Here : thousands × thousandths give units of thousands and units of thousandths.

This is a particularly important analysis when his work moves to the multiplication of a decimal number times a decimal number.

3. With a little sleight of hand, transform the green unit bead into 10 dark blue pills---tenths.

3. I have transformed the unit into 10 new little pills.
And they have changed color---now they are blue.
I have divided this unit bead into 10 parts, like the circle.



4. Show each of the tenths on one of the tenths of the circle, indicating equal value.

5. Explain the color blue of the tenths.

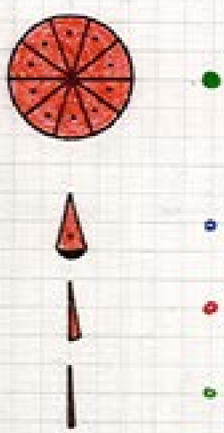
5. If I take 10 units together, what do I obtain? A ten. What color is the ten? The value of the bead that had the value of ten was blue: it was ten times greater than the unit.
Now this blue pill is one-tenth of the unit. It is ten times less than one unit.

6. Cut the cardboard tenth , and cut it into ten parts.

This fractional part of the tenth has a value of one-hundredth.

7. Transform one blue pill into ten red pills representing hundredths.
Show the cardboard hundredth and the red pill together.
Repeat the process for the thousandths; first cutting a thousandth from one of the cardboard hundredths and then transforming one red pill into 10 green thousandths.

If I take the blue pill and smash it, I will have ten hundredths.
This is one-hundredth---and this is one-hundredth. Why are they red?
If I take 100 units together, I have 100. The hundred is 100 times greater than the unit.
If I take 100 times less than the unit, I have the one-hundredth.
Now we are breaking the red hundredth into ten equal parts; and we get a pill with the value of one-thousandth.
One-thousand times the unit is one-thousand. It is green.
One-thousandth part of the unit is one-thousandth. It is green, too.



Presentation #2: The Symbols

Materials: the decimal system numeral cards. . .and a decimal point

1. Show the small numeral card 10---turn it upside down. Insert the point.

1. If I turn this numeral upside down and insert the decimal point, it is no longer a ten; but a tenth. It is still blue; the root is ten, but now it is a tenth.

2. Repeat with the hundred numeral and the thousand: the unit is the same even though it is upside down. Each time the decimal point is shown in a different position.

NOTE: This exercise is basically to point out the relationship of the tens to the tenths, the hundred to the hundredths, the thousand to the thousandths. It is NOT A LESSON IN WRITING DECIMAL NUMBERS.

10	0.1
100	0.01
1000	0.001

Presentation #6: Whole Number X Decimal Number: At the Second and THIRD LEVEL WORK

5,467 X 0.27 =

1. Second Level Work: The child proceeds with this new case as he has done the work on the second level for the first case. The main new element in this case is the position of the multiplier. The multiplicand is first shown on the board so that the last digit coincides with the last square to the right on the decimal checkerboard; that is, in this case the multiplicand must be shown on the top row where that last square to the right is the units square. Then the units of the multiplier must be shown on that square of the number strip which is horizontally adjacent to the row on which the multiplicand is shown. In this operation that unit digit is zero. Then, because the other two digits represent lesser hierarchies, they are shown below the unit on the number strip. The multiplication then proceeds as described for second level work, the child showing the product of each multiplication with only one bar on the actual product square and another bar on the next higher square to the left if there is a hierarchical carry-over.

2. The verbalization of each hierarchical multiplication continues to be an important element of the work. For the ~~first~~ second* partial product:

7 X 0.2 = 1.4	units X tenths = tenths
60 X 0.2 = 12.	tens X tenths = units
400 X 0.2 = 80.	hundreds X tenths = tens
5,000 X 0.2 = 1,000.	thousands X tenths = hundreds

* first partial product will be each digit of the multiplicand x 0.07.

3. At the conclusion of the multiplication of a horizontal row which represents the partial product, the child now writes the partial product. He writes each partial product WITHOUT THE DECIMAL POINT, and the guide for the positioning of that partial product is the color of the last square on which his partial product is shown, as in the whole number checkerboard.

4. As in his previous multiplications, the child obtains the product by sliding down all partial product bars on the diagonal, and reading the result after the reduction to one bar per hierarchy has been made. He writes the answer and checks that answer by adding the partial products he has already written.

5. An Important Observation: When we multiply the whole times a decimal fraction the product is less than the whole. This is an observation already made in the work with the real fractions (2 X 1/4 =) and on the decimal board work.

6. When the operation is concluded, we observe:
   We have multiplied whole by hundredths;
   and we obtain a whole and hundredths.

```
   5,467
 X  0.27
   38269     Decimal point
  10934      placed according
 1476.09     to product on board.
```

7. THIRD LEVEL WORK: Mental Carry-Over. As in the last passage of the work with the whole number checkerboard, the child now shows each multiplication as only one bar on that square which corresponds to the hierarchical product and carries over mentally any quantities which must be added to the next hierarchy. Then, when he multiplies 7 X 0.07, he shows the nine-bar on the hundredths square and mentally carries over four tenths for the next multiplication.

Presentation #7: Decimal Number X Decimal Number: Second and Third Level Work

NOTE: After the first case work on the first level, the child does the other work of multiplication on the second and third levels. As a new case is introduced, the work may again be at the second level, but the child quickly then moves to the third level of mental carry over.

23.25 X 43.7 =

The child, as in the previous operations shows his operation first on the mat with the decimal system numeral cards and the decimal number cards. He substitutes then the white cards for the multiplicand which he shows on the board, the last digit corresponding with the last square to the right on the checkerboard. He substitutes the grey numeral cards for the multiplier and then shows the multiplier on the number strip to the right. The new element here is that, once the unit has been placed to coincide with the horizontal row on which the multiplicand is shown, the tens digit is placed above (greater hierarchy) and the tenths digit below (smaller hierarchy.) The multiplication proceeds.

The multiplication always begins with the smallest hierarchy of the multiplier. Each partial product is formed on the horizontal row and written. Then the bars are brought down the diagonal, the total product read and the child checks that result with his addition of the written partial products. He shows the product of the multiplication with the symbols on the mat.

Presentation #8: The Design

The design for the decimal number checkerboard is constructed as in the work with the whole number checkerboard. Here we find out if the child has really understood the order of the hierarchies. In order to do the design work, the child must show his operation at the FIRST LEVEL OF WORK. He writes each multiplication and then represents that product as shown on the board in his geometrical design. His color guide is in the squares of the checkerboard.

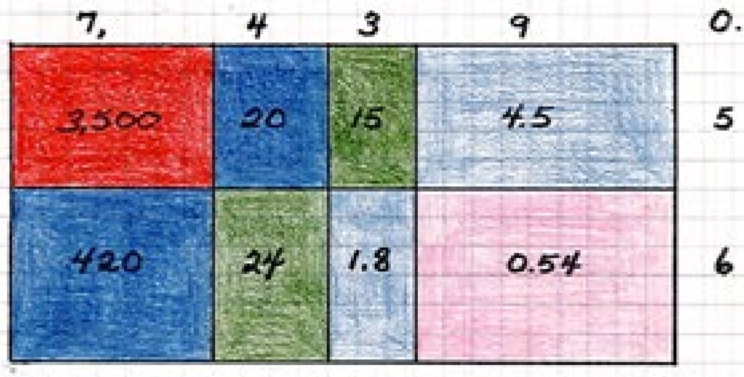
$$\begin{array}{r} 7{,}439 \\ \times\ 0.56 \\ \hline \end{array}$$

First partial product: bottom row

u × h^ths = h^ths 9 × 0.06 = 0.54
t × h^ths = t^ths 30 × 0.06 = 1.8
h × h^ths = u 400 × 0.06 = 24
th × h^ths = t 1,000 × 0.06 = 420

Second partial product: top row

1 × 0.1 = 0.1 9 × 0.5 = 4.5
10 × 0.1 = 1 30 × 0.5 = 15
100 × 0.1 = 10 400 × 0.5 = 200
1000 × 0.1 = 100 7000 × 0.5 = 3500



CONCLUSION NOTE: We have now shown the three cases of multiplication: 1) Decimal Number X Whole Number, 2) Whole Number X Decimal Number and 3) Decimal Number X Decimal Number; carried out with both the yellow decimal number board and the checkerboard for decimal numbers.

The emphasis of the work with the decimal board was: 1) the changing, 2) the relative value of each digit and 3) when the multiplier was a decimal number, the necessity of transforming it to a whole number before carrying out the operation.

With the checkerboard we emphasized: 1) the values obtained by multiplying a whole number and a decimal number in various combinations where no transformations were necessary and 2) the geometrical figure of multiplication.

With these materials, then, THE AIMS:

of the decimal board: to bring the child to understanding and the abstract carrying out of multiplication of whole and decimal numbers.

of the checkerboard: to bring the child to abstraction, through geometrical figures, in the multiplication of whole numbers and decimal numbers.

AGE: 8 - 9

DIVISION: Of Decimal Numbers

Here we consider four cases of division, each case examined with passages of increasing difficulty:

1) Whole number ÷ whole number: 3 ÷ 2 =
4 ÷ 3 =
2) Decimal number ÷ whole number: 0.25 ÷ 4 =
2.31 ÷ 3 =
3) Whole number ÷ decimal number: 3 ÷ 2.31 =
4 ÷ 0.25 =
4) Decimal number ÷ decimal number: 8.6 ÷ 4.3 =
0.40 ÷ 0.25 =

The child has encountered all of these simple divisions at the level of memorization.

DIVISION OF DECIMAL NUMBERS. . .

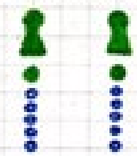
Material

1. The box of decimal discs and whole bead quantities in the hierarchical colors.
2. The numeral symbols of the decimal numbers and the whole numbers (small series from the bank game or the decimal system sets)
3. The box of green, blue and red skittles (from division.) Orange box.
4. A light green box containing a new set of "little skittles," in blue, red and green for the decimal number divisors. Miniatures.

Presentation: Case 1a: 3 ÷ 2 =

1. The child forms the dividend and the divisor with the num eral symbols and shows the operation on the mat with the operation signs.

1. I must divide 3 among 2.



2. Then he takes the quantity of beads to be divided and shows two skittles (green.)

3. He distributes one unit bead to each of the two; and then EXCHANGES THE REMAINING UNIT FOR TEN TENTHS DISCS. He distributes those tenths to the two skittles.

3. In our first work of division, we had a remainder of one, but now we can exchange that one remaining unit for ten tenths and our remaining quantity can be equally distributed.

4. We note that the answer, the quotient, is what the unit receives; and that now we have truly completed the division. The child shows the answer with the symbols on the mat and copies the operation.

4. Each unit has received one unit and five tenths.
The quotient is no longer a whole number, but a decimal number.
3 ÷ 2 = 1.5

Presentation: Case 1b: 4 ÷ 3 =

1. The child proceeds as in the first work. He shows the symbols for the operation, takes the corresponding quantity for the dividend and shows three skittles as the divisor. He distributes the units, one to each of the three skittles.

2. One unit remains. He exchanges that unit for ten tenths. And distributes the tenths, three to each of the three skittles. AGAIN ONE TENTH REMAINS.

3. He exchanges the tenth for ten hundredths. Again he is able to give each skittle three and one hundredth remains. He exchanges that hundredth for ten thousandths. . . and he begins to realize that this could go on forever.

4. CONCLUSION: We can conclude that not all quotients can be resolved, even with decimal numbers. There are some divisions which have no end. So we write the quotient: 4 ÷ 3 = 1.333.... (and so on. . .to infinity.)

NOTE: The children like to look for divisions with quotients that never end. . .to be sure they really do exist.

Presentation: Case 2a: 0.25 ÷ 4 =

1. The operation is shown with the symbols and the operation signs. The quantity is taken which represents the dividend (two tenth discs and five hundredths discs.) The four green unit skittles are shown.

2. The child knows that he must start by dividing the largest quantity, but it is not possible to divide two tenths among 4, so he exchanges, one tenth for ten hundredths. Then the second tenth for ten hundredths. AND HE DISTRIBUTES THE 25 HUNDREDTHS. One hundredth remains.

3. He exchanges that hundredth for ten thousandths---each unit receives 2 and 2 remain. He exchanges both those thousandths for ten thousandths (even though he may suspect that this division will continue on and on) and he discovers that now the twenty ten thousandths will distribute equally---5 to each of the skittles.

4. He reads what one unit gets: NO UNITS, NO TENTHS, SIX HUNDREDTHS, TWO THOUSANDTHS, FIVE TEN THOUSANDTHS. He shows that answer with symbols: 0.0625

DIVISION OF DECIMAL NUMBERS. . .

Presentation: Case 2b: 2.31 ÷ 3 =

. Here the two units cannot be distributed equally among three, so we must exchange the two units for tenths. Giving 23 tenths to be distributed. Each skittle receives 7 tenths.

With the remaining tenths we exchange for hundredths, giving us 21 hundredths for distribution. These will distribute equally to the three.

We read the quotient. Is is a whole number? NO. We have no units, seven tenths and seven hundredths. The quotient is 0.77 or SEVENTY-SEVEN HUNDREDTHS. That is what one unit receives.

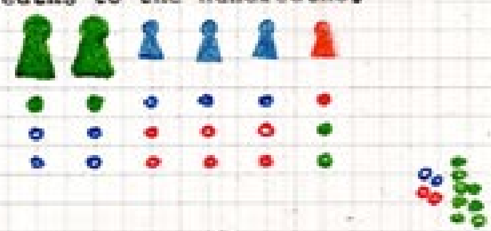
Presentation: Case 3a: 3 ÷ 2.31 =

1. Introduce the new box of "little skittles." Show the miniature blue for the tenths, the reds for the hundredths and the greens for the thousandths.

2. The child shows the operation with the symbols, takes the quantity in beads, and shows the divisor with two large green skittles, three small blue and one small red.

3. We begin the distribution by giving each of the unit skittles ONE UNIT BEAD. Note that each successively lesser hierarchy will receive ten times less than the previous. TWO EXCHANGES ARE THUS NECESSARY TO ACHIEVE THE ACCURATE DISTRIBUTION: the remaining unit bead for ten tenths and then one tenth for ten hundredths.

3. If we give units to units, then we must give the tenths ten times less. . .and the hundredths 100 times less. So we distribute tenths to the tenths and hundredths to the hundredths.



4. SECOND DISTRUBTION: The units receive tenths so the tenths must receive hundredths and the hundredths thousandths, necessitating another exchange.

5. HERE WE STOP: Now the child understands the hierarchical distribution. We read the answer as what the unit receives. . .and then we show both the quotient and the remainder with the symbol cards on the mat. 3 ÷ 2.31 = 1.2 r. 0.228

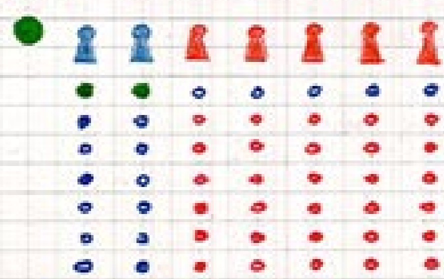
6. REVIEW THE VARIOUS QUANTITIES RECEIVED BY THE LOWER HIERARCHIES.

6. The unit receives 1.2.
What did the tenths receive? TEN TIMES LESS or 0.12.
What did the hundredths receive?

Presentation: Case 3b: 4 ÷ 0.25 =

1. Introduce the green disc to show the place of the units in the divisor.

1. There is now no unit in our dividend, but we must keep the place of the units.



2. The distribution is made, following the pattern as described in the previous cases.

3. What is the result? What the unit receives. If the tenth receives one unit and six tenths, then the unit will receive ten times MORE or 16.

4. IMPORTANT CONCLUSIONS:
NOTE: This work is parallel to the division of fractions. We see that if 1/10 or 0.1 receives 1.6, then one unit must receive that 1.6 plus (9 X 1.6) OR (1.6 X 10).
AND
We can recall the ways we obtain the division quotient in the written fraction operation: 4 ÷ 1/4 = 16

$$4/1 \div 1/4 = \frac{4 \times 4}{1 \times 1} = 16$$

4. We must give the same amount to the other fractional parts of the unit in order to show the quotient which is what the unit receives.

When we are dividing a whole number by a decimal number, what are we doing?
4 ÷ 0.25 = 16
A multiplication.

DIVISION OF DECIMAL NUMBERS. . .
Case 3b: 4 ÷ 0.25 = 16. . .

4. . .CONCLUSIONS: We can show a comparison between the multiplication of a whole number by a fraction (or decimal number) and a division of a whole number by a fraction (or decimal number.)

$$10 \times ? = 5 \quad 10 \times \frac{1}{2} = \frac{10 \div 2}{1 \div 1} = 5$$
$$8 \div ? = 32 \quad 8 \div \frac{1}{4} = \frac{8 \times 4}{1 \times 1} = 32$$

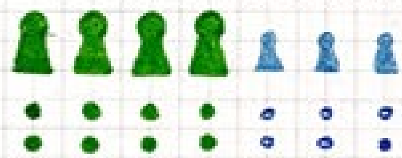
We also note that we are, in the division of 4 by 0.25 performing a group division.

4. . .In the multiplication of a whole number by a fraction or decimal number, we are actually dividing.
And in the division of a whole number by a fraction or decimal number, we are actually multiplying.

When we divide 4 by 0.25, we are asking: How many groups of 0.25 are contained in 4?

NOTE: We can conclude that multiplication and division must be given parallely in the work of fractions and the work with the decimal numbers. And we must utilize the centesimal circle as a passage, to relate the fractions and the decimal numbers as the child does the operations in both works. Thus he encounters the same situations and clearly can relate the fractional value of the actual fractions and the decimal numbers and decimal fractions.

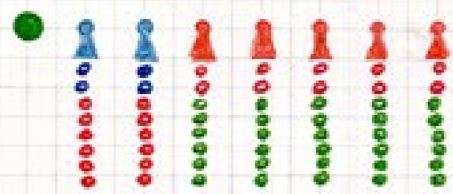
Presentation: Case 4a: 8.6 ÷ 4.3 = 2



Here we see that both the dividend and the divisor are decimal numbers, but the quotient is a whole number.

Presentation: Case 4b: 0.40 ÷ 0.25 =

Again the disc showing the place of the units is utilized.
It is useless to take 40 hundredths since we know that 40 hundredths is equal to four tenths. . .so we begin the distribution with tenths to the 10ths and then exchanging, hundredths to the hundredths.
The answer, what one unit receives, must then be ten times what the tenth receives:
0.40 ÷ 0.25 = 1.6



If the tenths received one tenth and six hundredths, then the unit will receive one unit and six tenths. 10 times more.

NOTE: Now we must make the work for the child easier, offering a different means which is closer to abstraction. He knows that when we divide a whole number by a decimal number, we may apply the invariant property which says that: when we multiply or divide the dividend and the divisor by the same number, the value of the operation (the division, or the fraction) does not change.

THE INVARIANT PROPERTY: A Movement To Abstraction

We have, to this point, done rational arithmetic. Now we must move the child to abstraction where all of these processes are not necessary. We utilize the invariant property which the child should have thoroughly understood in his work with fractions. This invariant property enables us to transform the divisor, when it is a decimal number, to a whole number, by multiplying it times 10 or times 100. . .and at the same time multiplying the dividend by 10 or 100. . .

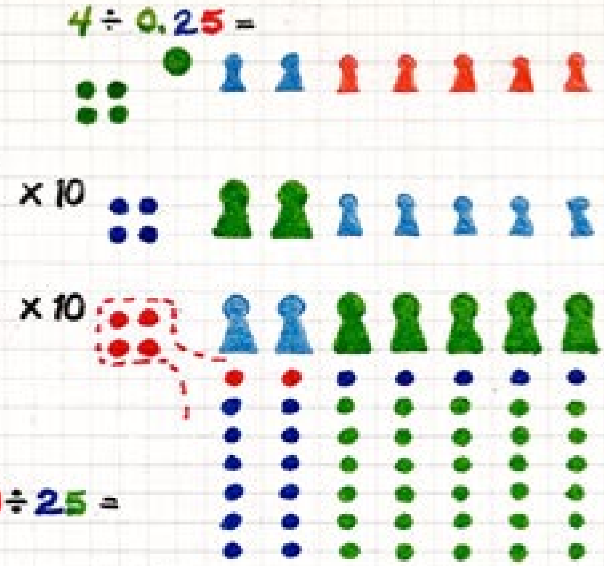
The rule is that if we multiply or divide the two by the same number (of zeros), the value of the division is not changed, just as the value of the fraction is not changed. If there is a remainder, that number also must be multiplied or divided by the same number.

We can approach the invariant property in three ways:

A. The Invariant Property shown with the Quantities and Skittles. 4 ÷ 0.25 =
The child knows that he can multiply the dividend and the divisor by the same number without changing the value. And he has already done this operation with the quantities and skittles in the long method as described in Case 3b. He knows that the result is 16.

Division of DECIMAL NUMBERS. . .

A. The Invariant Property. . .



1. Show the operation with the symbols; then show the quantities and the skittles.

2. Multiply both the divisor and the dividend by 10---thus changing the hundredth skittles to tenths, the tenths to units AND the quantity to tens.

3. Multiply again by 10---we have now multiplied both divisor and dividend by 100. AND THE RESULT IS A DIVISOR THAT IS A WHOLE NUMBER.

4. Replace the symbols with the new symbols which represent the new divisor and dividend shown.

$400 \div 25 =$

5. Now the child can, as an additional exercise, distribute the quantities OR HE MAY NOW CARRY OUT THE DIVISION ABSTRACTLY. .

```
     16
25 ) 400
     250
     150
     150
```

16

6. We compare the results---of the first operation done with the quantities and skittles and that done abstractly: they give the same result.

B. The Invariant Property shown with the Yellow Decimal Number Board $4 \div 0.25 =$

1. Again begin by showing the symbols for the operation.

2. Then take the dividend: 4 and show that numeral card at the top of the units column on the yellow board. Then take the double grey zeros card (two zeros) and multiply the 4 by 100 by showing the grey card to the right of the 4 digit, thus sliding that 4 two places to the left. SUBSTITUTE THE 400 symbol card for the new quantity as shown. Replace the symbol in the operation.

3. Multiply the divisor by 100 in the same way, placing the numeral cards over the corresponding columns at the top of the decimal number board. Then adding the two grey zeros---that is, multiplying by 100, thus moving the digits two places to the left. AND SUBSTITUTE THE NEW NUMERAL CARDS: 25 Replace in operation.

4. The result is a divisor that is a whole number. And the operation can now be carried out abstractly.

C. The Invariant Property shown as a Calculation with the Decimal Fraction.

1. IN THE CHILD'S NOTEBOOK: he writes

$$4 \div \frac{25}{100} =$$

1. We can write 0.25 as a decimal fraction: 25/100.

2. $\frac{25}{100} \times \frac{100}{1} = 25$ THEN $400 \div 25 =$

$$4 \times 100 = 400$$

2. How do we transform 25/100 into a whole number? We multiply by 100. Therefore, we must also multiply the dividend times 100.

3. Conclusion: Here we have transformed the divisor into a whole number by using the decimal fraction. And our result will be the same because we have not changed the value of the division.

4. The child does the division abstractly.

REPEAT THE TRANSFORMATION EXPERIENCES IN ALL THREE MODES, USING EACH OF THE EIGHT CASES.

A Special Case: Considering the Remainder Case 3a: $3 \div 2.31 =$

If we apply the invariant property to this operation, we must remember the rule which says that if we multiply the divisor and the dividend times a certain number, we must also multiply the remainder.

DIVISION OF DECIMAL NUMBERS. . .

THE Case of the Invariant Property with a Remainder. . .

$3 \div 2.31 = 1.2$ r. 0.228 . . . our first result.

$3 \times 100 = 300$ and $2.31 \times 100 = 231$, so we can divide $300 \div 231$

```
      1.2   r. 22.8
231 ) 300.0
      231
       69.0
       46.2
       22.8
```

. . . but we must also multiply our remainder × 100 . . . thus

$0.228 \times 100 = 22.8$, the remainder we show with this division.

The remainder is that fractional part of the dividend which has not been distributed . . . or divided . . . among the units of the divisor. It is, then, a fraction where the numerator is the remainder and the denominator the divisor:

$$\frac{\text{remainder}}{\text{divisor}} \text{ OR } \frac{22.8}{231} \text{ which equals } \frac{0.228}{2.31}$$

Therefore, our remainder can be shown in either form, according to the operation, and the value is unchanged.

$3 \div 2.31 = 1.2$ r. 0.228 $=$ $300 \div 231 = 1.2$ r. 22.8

NOTE: from Montessori: In the traditional school, the work starts with the invariant property. For us it is the point of arrival. For us it is a rational consequence arrived at through previous experiences. Introduced here, it simply helps the child to simplify his work.

NOTE: If all the work is done with the decimal numbers, particularly that work on the yellow decimal number board, the children have a strong preparation for understanding the metric system.

AGE: 8 - 9

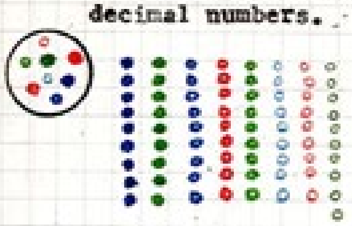
DECIMAL NUMBERS. . .

Presentation #3: The Quantities II

1. Begin with one red bead, 100, and exchange 1. for 10 blue tens. Then change the blue ten for 10 green units.
NOTE: the child knows the exchange process to this point.

   1. I have one hundred. I want to change it for ten tens.
   Now I want to change one ten for ten units.

2. Now we change the unit for 10 tenths. Then exchange one tenth for 10 hundredths. . . and proceed through all the position of the decimal numbers. Display.



   We can exchange this unit, too---for ten tenths.
   And we can exchange one tenth for 10 hundredths.
   The ten-thousandths are a lighter blue, the hundred-thousandths are a lighter red, the millionths are a lighter green. They are a lighter shade of our hierarchical colors to remind us that their value is less that that of the last three groups. This is the class of the millionths.

3. Bring a golden circle from another table: take the unit to another position away from the displayed group and crown the green unit bead as the center.

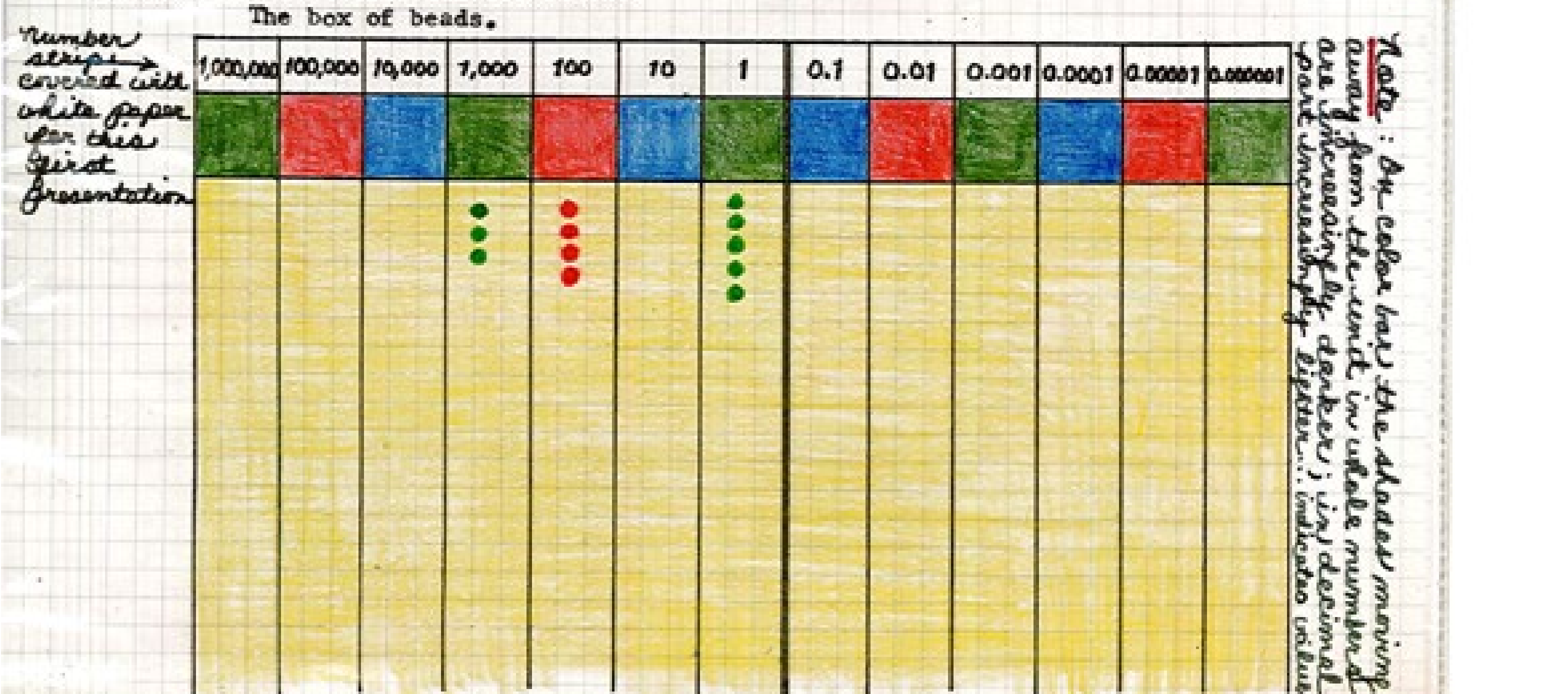
   2. This unit is the center of our work.

4. Alternately place the whole quantities to the left of the unit and the decimal quantities to the right, taking the whole quantity beads from the box and the decimal quantities from the displayed group. Note as they are placed in pairs (tens/tenths) that the color is the same. Review the concept of 10 times greater and ten times less in relationship to the unit. (The light blue pill is 10,000 times less than the unit---it is one ten-thousandth; but the blue ten-thousand bead is ten-thousand times greater than the unit.)

5. CONCLUSION: The million bead is a long way from the unit, far to the left, but it has a million times more balue than the unit. The millionth is far away from the unit to the right, and it has a value one-million times less: it is one-millionth of the unit. The farther away from the unit to the right a cipher gets, the more its value decreases. The farther away from the unit to the left, the greater the value.

EXERCISES: Formation and Reading of Quantities on the Board

Material: The decimal numbers board:
The box of beads.



EXERCISES: Formation and Reading of Quantities on the Board. . .

1. The teacher takes a certain quantity of beads (3 green thousand beads, 4 red hundred beads, and 5 units) using first only the whole quantities and making use of the positional value in the way in which we show the beads displayed. The child names the quantity shown and then places that quantity on the board, following the color coding at the top and the dark black line which we note is the division between the whole and the fractional parts of the unit. The placement here will be all on the left side of the board.

2. AFTER SEVERAL EXAMPLES WITH ONLY WHOLE NUMBERS: The teacher forms quantities which include the decimal number pills. The child names the quantity they represent. He shows them on the board. INCREASE THE DIFFICULTY.

3. Teacher says the quantity, the child lays out the corresponding materials and then shows the quantity positioned on the board. GRADUALLY INCREASE DIFFICULTY, INCLUDING ZEROS: 1,000,001.0001.

4. Child chooses the quantity from the materials, places that quantity on the board and reads the value.

GAMES: The teacher requests a quantity which requires an exchange. . .
"Take 13 tenths and place them on the board."
"Now I want these 13 tenths in another way."

NOTE: The child needs many such exercises to reinforce the concepts of the decimal numbers. He must understand the hierarchy, and the succession of changes possible.

Presentation #4: The Symbols II

Materials
1. All the symbol cards from the BANK GAME. . .in the box. Units to millions.
2. A new series of symbols for the decimal numbers: 0.1 - 0.9; 0.01 - 0.09; 0.001 - 0.009; 0.0001 - 0.0009; 0.00001 - 0.00009; 0.000001 - 0.000009. In the lighter shades of the hierarchical colors.

1. Lay out the unit symbols 1 - 9.
Lay out the tens.
Lay out hundreds.
Proceed through the millions, always noting the color and the movement to the left.

1. What is to the left side of the units? What color? What is to the left of tens? What is to the left of hundreds?

2. Introduce the new series of symbols, laying out one hierarchy at a time to the right of the unit, noting the color and the movement to the right.

2. What comes to the right of the unit? What color is it? Why?

3. Review the layout when all symbols are placed.

	1000	100	10	1	0.1	0.01	0.001	
	2000	200	20	2	0.2	0.02	0.002	
	3000	300	30	3	0.3	0.03	0.003	
.....	4000	400	40	4	0.4	0.04	0.004	....
	⋮	500	50	5	0.5	0.05	⋮	
			60	6	0.6			
				7				

3. The units are in the middle. To the left are the whole numbers which go to the millions and to the right are the decimal numbers which go to the millionths.

4. Take the numeral 1 aside, to another table, crown it with the golden circle and place the single digit cards alternately to the left and right. Compare the pairs laid down according to color and value.

4. Because the unit is the center of the whole work, we crown it.
What is first to the left of 1?
What is to the right?
They both have the same color. the same value? NO. The ten is 10 times larger; the tenths are only one-tenth part of the unit. BECAUSE THEY HAVE DIFFERENT POSITION

4. Crowning the unit. . .

10,000 1000 100 10 (1) 0.1 0.01 0.001 0.0001

We have examined here all the hierarchies from the millions to the millionths with the unit as a reference.

4. The composition of the number is the same: the ten is composed of a zero and a one; the tenth is composed of a zero and a one, but the arrangement is different. THE VALUE OF THE QUANTITY DEPENDS ON ITS POSITION. . .IN RELATION TO THE UNIT.

5. Present the decimal system board again, now revealing the top row where the ciphers are shown to correspond with the hierarchical colors. EMPHASIZE: If we move towards the units from the millions, that is left to right, the value decreases by ten times (powers of ten) with every hierarchy. Moving from the millionths towards the units, that is right to left, the value increases with every hierarchy. AND in the whole numbers, the numeral card size increases with an increase in value; in the decimal number numeral cards, the value decreases as the size of the card increases.

EXERCISES: The Formation and the Writing of the Symbols: All symbols displayed

1. The teachers forms the number; the child reads it and forms it on the board. Here we begin with whole number combinations. The child combines the ciphers shown as in the magic game to form the number and then shows that number on the number strip at the top of the board. THEN we introduce three cards of decimal numbers and the child discovers that the magic game must be done in reverse. The cards now are pushed to the left. He reads the number: two tenths, one hundredth, 6 thousandths. FINALLY we take a combination of whole and decimal numbers: the child forms first the whole number with the ciphers, then the decimal number with the reverse magic game, and finally he superimposes the units on the zero of the tenths to form the whole number and places it on the board.

2. The teacher says the number and invites the child to form it with the symbols, then to show it on the board, reading it.

3. The child forms the quantity with the symbols, reads it and shows it on the board. NOTE: In each case, he records the quantity in written symbols in his notebook.

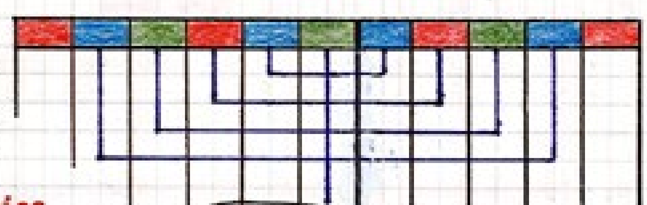
NOTE: On Reading the Numbers

There are two stages of reading the numbers: the first one is noted above, where the child reads the numbers as tenths, hundredths, etc. . . THEN we introduce the correct reading of the number, giving the decimal number the name of the last digit. SO. . . 253.637 is two-hundred fifty-three AND six-hundred thirty-seven thousandths. Here the child must unite the sounds and do the work of synthesis as in the movement from the movable alphabet sounds to the whole word.

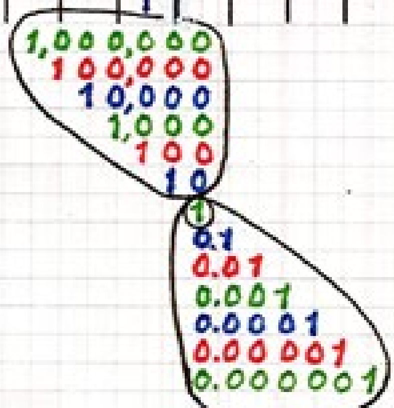
In reading whole numbers we give our emphasis to the first digit; in the decimal numbers we emphasize the name of the last cipher which names the whole decimal part.

A Preparation for Uniting the Symbols and the Quantities: The Concept of Unit

1. Use a chart which will fit onto the board, with the image of a candellabra, pairs of arms connecting the tens and tenths, the hundreds and the hundredths, the thousands and the thousandths, etc. The squares are like the flames. The center post is the unit. The child copies the formation in his notebook.



2. With the numeral cards, show the unit as the center, then build up and down with the whole numbers and the decimal numbers. The child copies this, circles the resulting figure. Then he can cut it out, and the result is a kind of propeller that spins on the center---the unit. A chart depicting this is an interesting wall chart.



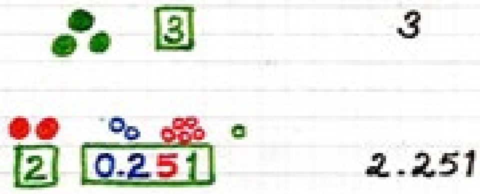
Presentation #5: Matching the Quantities to the Symbols

Material: The decimal numbers box of materials, the board, all the symbol cards.

1. Lay out all the symbol cards. . . in correct positions.

   These are symbolic quantities. They take their value from the position they occupy.

2. Teacher shows a quantity with the materials (using whole numbers first). The child reads the value and matches the symbol. Then he shows both on the board and reads it.
   GRADUALLY INCREASE THE DIFFICULTY.



3. Teacher says the quantity, the child takes the quantity first, then matches the symbol card, forms the number and places both on the board, reading it. VARIATIONS prove good exercises in exchanging.

   VARIATION: Give me 10 tenths. (The child can take 10 tenths, but he cannot locate the symbol card; he discovers he must exchange for one unit.)

4. Child thinks of a quantity, shows the quantity and the symbols, then forms both on the board.

   THEN GIVE THE SYMBOL FIRST. . . . . .

5. The teacher forms the number with the symbol cards, the child takes the quantity; he reads it and places both on the board.

6. Teacher asks for the corresponding symbol cards for a number which the child finds and forms together; then he takes the quantity and places both on the board, reading.

7. The child does all the work. He thinks of the number, takes the symbols, the quantity shows both on the board and reads it.

GAMES: Reading and Writing Decimal Numbers

A. Quantities without Symbols: Who Has More?
   The teacher asks one child to take one unit (quantity); another to take four tenths. Each lays out his quantity. The teacher asks: "Who has more?" "You have four and he has only one BUT four parts of the unit, 4/10, has less value than one unit."
   VARIATION: 2 units/20 tenths: Both have the same quantities; the same values.
   VARIATION: 1 unit/8 tenths: Who has more? How much more? Two tenths more.

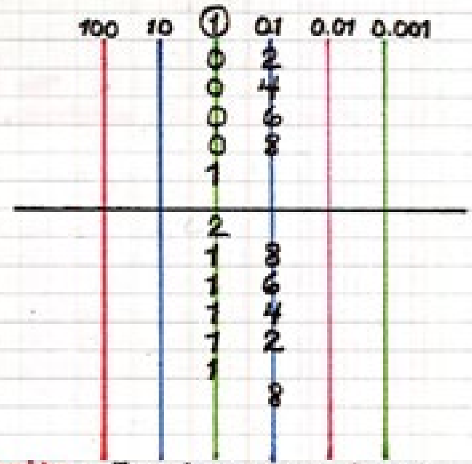
B. Symbols without Quantities: Who Has More?
   Each child gets a symbol and the comparison is made as in the above game: 1,000,000/0.000001 : Who has more? Both parts have the same digits, the same size, BUT the difference between them is 999,999.999999

C. Cards and Quantities: Same Game

NOTE: The children need much practice of this kind in the concepts of which numbers have more and less value. Also in the changing one makes in working with the decimal numbers. And in the value according to position.

D. Progressive and Regressive Counting with Quantity and Cards:
   1) Progressive: the child makes the form, as shown, in his notebook, color-coding the lines. Then he shows on the board a certain number of tenths (2) and increases each time by that amount. When he comes to 10 tenths he changes the quantity on the board and notes the change on his form by moving over a line.

   2) Regressive: Using the form he takes away 2 tenths or 5 tenths or 3 tenths at a time. Here he begins with whole units on the board; and must immediately make the exchange.
   INCREASE DIFFICULTY: Take 2 hundredths from 2 units. Two changes required.



E. Dictation and Reading Written Quantities

1) The teacher dictates a quantity: the child writes on the lined form he has prepared in his notebook. GRADUALLY INCREASE THE DIFFICULTY:

235.418 ------ 24.008

2) Teacher prepares slips of paper with quantities written in words: the child reads them and writes them in his notebook. CONTROL: the teacher asks the child to read the number he has written.

NOTE: **On Progressive and Regressive Counting:** this exercise, as noted in game D, is first given as an exercise strictly with the quantities. . . .using the form the child composes in his notebook. **The second period of the game** utilizes the symbols, the child showing the progressive or regressive counting exercise only with the numeral cards. **The third period is the culmination** in which the child combines the use of the symbol cards and the quantities. Both the second and third periods can be an effective collective lesson; a quiz.

ADDITIONAL STRIPS FOR THE BOARD

There should be prepared two additional strips for the top of the board: the first used specifically in work with the decimal numbers as decimal fractions. Introducing this strip, we show the 10th fractional part of the circle inset in the position of the tenths on the board. **1/10 is one-tenth of the unit.**

1,000,000	100,000	10,000	1,000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$	$\frac{1}{10,000}$	$\frac{1}{100,000}$	$\frac{1}{1,000,000}$

In the second additional strip we use the powers of ten both for the whole numbers and the fractional notation. This strip is used whenever we want to emphasize the powers of ten in connection with the hierarchies. The child has already encountered the powers of ten with the materials of the hierarchy and in his work with the chains.

Both strips recal the ways in which the decimal numbers can be written without changing the values, an important impression.

10^6	10^5	10^4	10^3	10^2	10^1	1	$\frac{1}{10^1}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$	$\frac{1}{10^6}$

AN IMPORTANT COMPARISON BETWEEN THE WHOLE NUMBERS AND THE DECIMAL NUMBERS WITH RESPECT TO THE UNIT: A Group Exercise

	WHOLE NUMBERS	(1) DECIMAL NUMBERS
Numeration	⟶ Regressive (Higher value to lesser)	Progressive ⟵ (Lower value to higher)
Color	⟵ darker (movement away from unit)	lighter ⟶ (movement away from unit)
Formation symbol cards	The magic game is from left to right.	The magic game movement is from right to left.
Superimposition overlapping of cards (symbols)	Card with the greatest value is on the bottom.	Card with the greatest value is on the top.
Length (of the symbol card)	The longer card has greatest value.	The shorter card has greatest value.
Writing	Number is written beginning with the unit, least value and moves to left.	Number is written beginning with tenth, greatest value, movement to the right.

134,789.434567
⟵ ⟶

Value	Beads represent n X 1. (100 X 1 : 100+)	Pill represents fractional part of 1. (1/100 of 1 : 100 −)
Nomenclature	Hundreds (ten tens)	Hundredths (ten tenths)

II. Operations: Addition, Subtraction and Multiplication with the board.

Presentation #1: Dynamic Addition

Material

1. The yellow board.
2. Box of beads and discs.
3. The bank game numeral cards and the new series of decimal number cards, ALL LAID OUT IN PROPER ORDER ACCORDING TO HIERARCHY.

NOTE: In our work we are not interested in working with large whole numbers. Therefore we usually limit that whole part of the number to three digits. But in the decimal numbers used, we give the operations with a variety, composing the decimal numbers with zeros and increasing the difficulty to include more digits.

1. The teacher presents the problem on a slip: 123.756 + 89.877 =

2. The child copies the operation in his notebook: NOTE: Emphasize that in writing the decimal numbers, it is again to line up the hierarchies carefully in columns. This results in a straight column of decimal points.

```
  123.756
+  89.877
```

3. The child forms both addents with the symbol cards; and then shows the whole operation on the mat with the operation signs.

```
[+] 123.756
     89.877
```

4. He forms the first addent on the board with the beads and pills; then, leaving a short distance on the board below the first addent, he forms the second addent.

5. He then executes the addition by combining the quantities shown on the board, beginning with the column at the farthest right; that is, the column of the lowest hierarchy. When the child arrives at ten of any hierarchical material, he must exchange those ten for one of the next larger hierarchy, and show that substituted quantity in its proper order. . . .the next column to the left.

6. When the combining is completed, the child reads the answer from the quantity shown. Then he takes those numeral cards to show the sum and places it below the operation shown on the mat. He copies that operation in his notebook. NOTE: Now the numeral cards he is using are all of the same size.

NOTE: The child should actually count the discs and beads to reach ten and then do all the necessary exchanging with the materials to reinforce the concept. The child reaches abstraction quickly, but he must master the changes. Therefore, additions with many changes are important.

EXERCISE: 0.999,999 + 0.000,001 =

Presentation #2: Dynamic Subtraction

1. The teacher presents the operation on a prepared slip: 304.307 - 197.149 =

2. The child writes in his notebook:

```
  304.307
- 197.149
```

3. He forms the minuend and the subtrahend with the corresponding numeral symbol cards, then shows them on the mat with the operation symbols.

4. The child forms now the <u>minuend</u> on the board with the corresponding quantities.

5. He begins the operation of subtraction with the hierarchy of least value. Here this first subtraction of the thousandths from the thousandths indicates a double change. We must take nine thousandths from seven thousandths. So we must exchange first one tenth for ten hundredths, then one hundredth for ten thousandths and finally we can take away the nine thousandths.

6. Proceed right to left hierarchically across the board, taking the necessary subtrahend. SHOW THE MINUEND IN LITTLE PILES TO THE SIDE OF THE BOARD***WHAT HAS BEEN TAKEN.

7. When the subtraction is completed, the child reads the answer from the quantity left on the board---the difference. He forms the difference with the numeral cards and shows that answer on the mat below the operation. He copies it in his notebook.

8. NOW HE CHECKS the operation: showing the collected ~~minuend~~ subtrahend which he has put to the side on the mat in hierarchical piles on the board below the quantity now shown---the difference. He does the addition to achieve the minuend. WHEN THE PROOF HAS BEEN MADE, HE CHECKS THE RESULTING QUANTITY WITH THAT SUBTRAHEND SHOWN WITH THE NUMERAL CARDS ON THE MAT.
NOTE: It is particularly important that he make this proof because it is good word in making the changes.

EXERCISES: 0.999,999 - 0.000,0001 =

Prepared operations of increasing difficulty that require many changes and include the zero in various positions:

3.000 - 0.123 =

1. - 0.000001 =

1. - 0.999,999 =

Presentation #3: A Preparation for Multiplication

Material
1. A green metal inset frame of the whole circle (which looks very much like the Montessori protractor) called the centesimal circle because it is divided into 100 parts. It is the "decimal circle." (centesimal means 1/100) On it the tens are numbered; a line is traced from the center to the zero.
2. The circle inset divided into fractions: the whole and the tenths, the halves, the fourths, the fifths, the thirds, the ninths. . .
3. A cardboard circle.

Presentation: The Transformation of An Ordinary Fraction to a Decimal Fraction to a Decimal Number

1. Introduce the centesimal circle. Take the whole circle inset and fit it into the frame, noting that it covers all the surface of the circle frame.	1. This circle has been divided into 100 parts. So we can say that 1 equals one-hundred hundredths. . . $\frac{100}{100}$ This circle equals one-hundred hundredths. $\frac{100}{100}$
2. Taking the whole divided into halves, fit both parts into the centesimal circle, noting congruency. Then REMOVE ONE HALF and read the number where the diameter shows 50. Introduce notation ---from fraction to decimal fraction to decimal number.	2. THEN one-half the circle equals 50 hundredths. We want to find out how this is written as a decimal fraction. SO $\frac{1}{2} = 50/100$ How can we write 50/100 as a decimal no.? $\frac{1}{2} = \frac{50}{100} = 0.50$
3. Use the 1/4 inset, this time placing the inset piece vertex to center and side to line, as the child has done with the Montessori protractor. The child reads the fractional indication, writes the transformations.	3. 1/4 is the ordinary fractional value of this fourth of the circle. We read 25. So this corresponds to 25/100 OR $\frac{1}{4} = \frac{25}{100} = 0.25$
4. Replace the 1/4 in the inset frame.	4. This is 1/4 or 25/100. So it must be contained in the whole 4 times.
5. Take 1/5. Then 1/3. Then 1/9.	$\frac{1}{5} = \frac{20}{100} = 0.20$ $\frac{1}{3} = \frac{33}{100} = 0.33$ $\frac{1}{9} = \frac{11}{100} = 0.11$