THE FRAMES OF HIERARCHY: The passage from real quantities to symbolic hierarchical quantities.

Introduction

In the decimal system, the objects which represented the ten, hundred and thousand, even if in the order of bar, square and cube, were actually made up of the quantity 10, 100, or 1000 units. NOW ten, hundred and thousand are represented by one bead. Each of the hierarchies is represented by a single bead. There is a logical, internal development in which the progression of numbers of objects no longer corresponds. We pass from real quantities to symbolic hierarchical quantities.

Our bead corresponds to 10 of the inferior order and to one-tenth of the superior order.

In multiplication, there are two difficulties: the memorization of the products and the rapid location of the position of the ciphers of the numbers. The value of each cipher depends on the position which is occupies. In the work we now meet, the most important consideration will be the position of the digit of the number: the value of the digit is relative to the position and no longer to the real quantity.

The latter is the Absolute value; the first, the value as determined by position, is the Relative value.

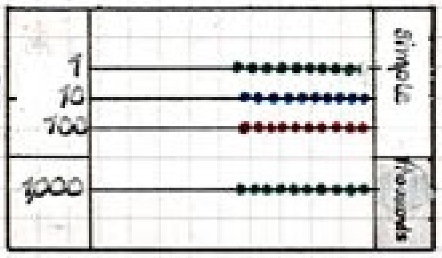
Every digit has an absolute value of the simple units independent of the position it occupies in the number; and it has a relative value dependent on the position.

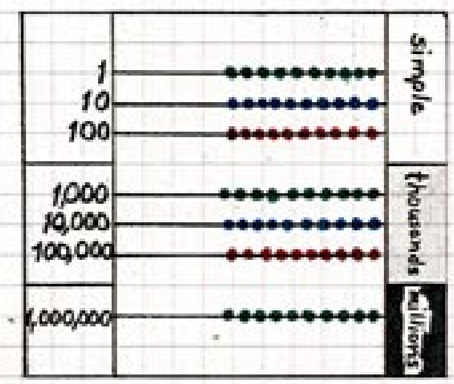
Montessori makes an analogy with our society composed of the president, the ministers, the senators, the deputies, and the simple citizen. As men, they are all equal. But what distinguishes them is the different social position they occupy in the organization of the state. And so with the ciphers: they can be humble units or millions, depending on their position.

We now have a piece of equipment which identifies not by quantity, but by position. This is the bead frame. The quantities now are symbolic. With the decimal system materials, we may display quantities in any position. The cube is still the cube, always representing 1000 wherever it is positioned. The child, therefore, can always recognize the ten, the hundred, the thousand.

With the bead frame, we introduce the concept of hierarchy. These quantities are reduced to symbols and thus, they must have their own exact place. Only in the correct place do they hold the decimal numeration. Therefore, they are fixed on the frame, always in the same position.

We are abandoning certain sensorial experience: numerosity, size, weight. And we must introduce a new quality: color. It is the necessary element for the recognition and distinction of the quantity. Most important is that the new hierarchical representatives must maintain, without variation, their position. The three colors which distinguish the order are the green (units), blue (tens), and red (hundreds) which the child has already met with the same significance in the symbols for the decimal system and in the stamp game.



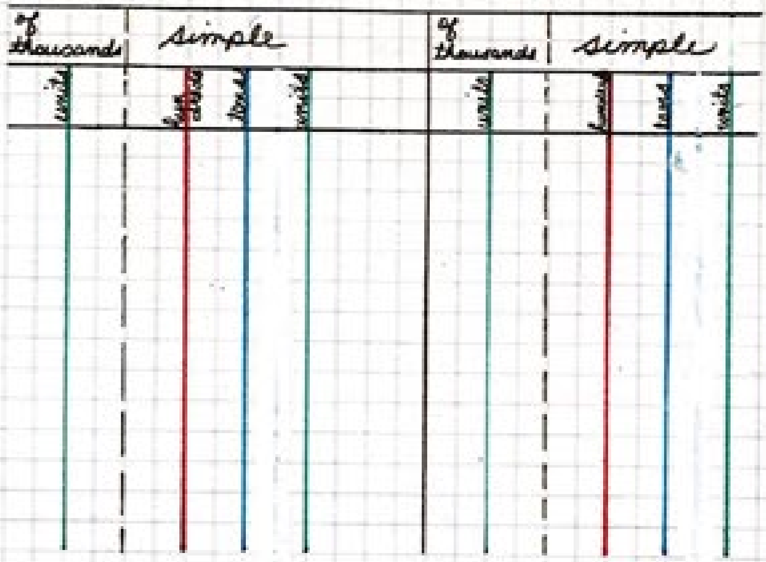


The bead frame resembles the ancient Chinese abacus, still used today. Such calculation began as simple lines drawn in the sand, and pebbles used for the calculation on those lines. "calculus" comes from the Latin word for pebble.

The small bead frame has only 4 wires, going to 1000; the larger one to the class of millions. On the first there is a larger space between the third and fourth wires, indicating the position of the comma and the passage to the next class. In the larger frame, we see this space again after the sixth wire. The symbols are written on the left; on the right the names of the classes are given.

THE BEAD FRAMES. . .

The frame is accompanied by forms, lined sheets divided into two parts by a black line and reproducing the bead frames two times.



Presentation

1. Position all the beads to the far left. Begin by showing the golden unit bead and showing its corresponding bead on the first wire. As each golden bead is shown, move one green unit bead across the frame to the right.	1. What is this? One unit. This golden bead corresponds to this green bead. And also this one corresponds to one golden bead, and this one. . . .
2. Note the change at 10. Then show the blue beads as corresponding to ten-bars in the same fashion as above with the units.	2. When we got to 10 with out golden beads, did we take 10 loose beads? We took one bar. This bar of 10 corresponds to the blue bead. . .and this one corresponds to one ten. . .and this one. . .
3. Show the correspondence of squares to the red beads on the third wire, again moving the beads across one at a time as the squares are shown.	3. This BIG square of 100 corresponds to this one little red bead. . .and this one. . .
4. Show the cube as corresponding to the green bead on the last wire. Then move all the beads back to the left side of the frame.	4. And when I get to ten 100s, what do I change it for? Look! This whole 1000-cube corresponds to this one green bead. . .and this one.
5. Second period lesson with 1 unit and 1 thousand.	5. Now I want to see if you have understood. Show me what this golden bead corresponds to hierarchically. And this 1000-cube?
6. Give the concept of position with those two green beads at the right of the frame and two single green beads in hand.	6. I'm going to play a little joke on you. On this frame, what do you see? I have in my hand two green beads. What are these worth? 2 units. But what about these? This is worth 1000 and this is worth 1. It is the position which gives this symbolic quantity its value.

THE BEAD FRAMES. . .

EXERCISE #1: Numeration on the basis of position: Reinforcing the importance of the number's position.

DIRECT AIM: To bring to the child's consciousness the function of the position: the relative value of the digits.

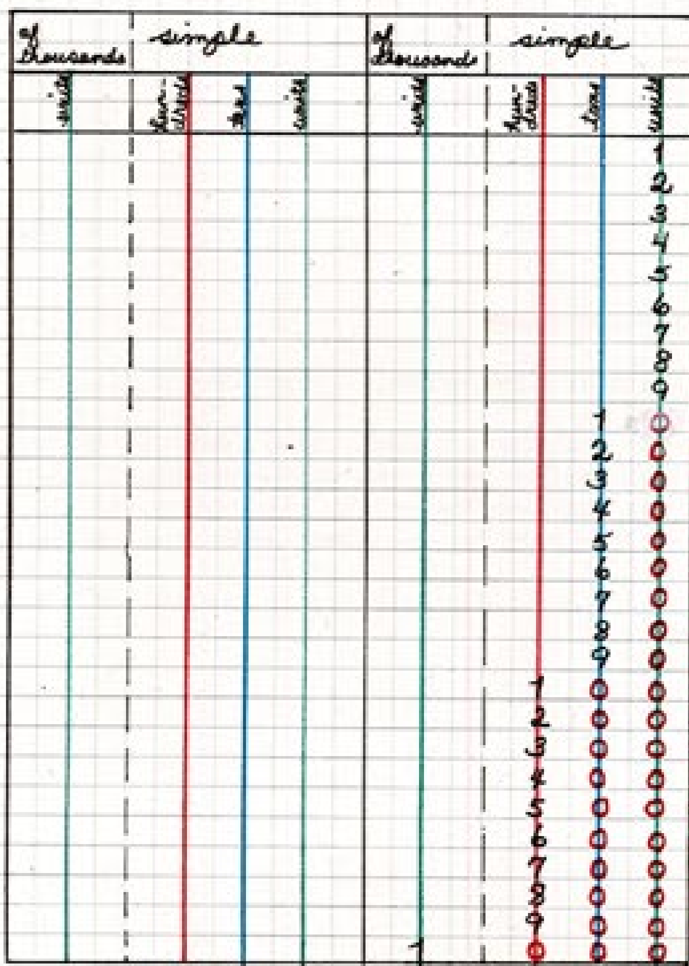
Presentation

1. Introduce the printed form for the first bead frame:
   Turn the form sideways first so that the child can see how the colored lines correspond to the wires of the frame. He notices that the frame is reproduced twice on the form.
   Note the dotted line which marks the place for the comma; note the corresponding space between the third and fourth wires on the frame. . .also the passage here from one class to another.

2. Take the bead frame, and count first the units, then tens, hundreds, to three thousand. For each bead moved (right to left) across the frame, show the digit on the form. WHEN THE TENTH BEAD IS REACHED ON ONE WIRE, SHOW THE EXCHANGE OF THOSE TEN FOR ONE OF THE NEXT HIERARCHY AND BEGIN THE NUMERATION OF THAT NEXT CLASS.

2. This is 1.
   We write a 1 on the first green line---that line is our simple units column.
   This is 2.
   We write 2. . .
   When I have this tenth bead,
   I have one ten.
   So I must exchange these ten units for one ten on the second wire.
   And I write 1 on the second line---the blue line which shows our simple tens.

3. When the counting and notation reaches 1000, the child has reached the last line of the form---only 28 lines.



This is 900. . .
And with one more hundred bead, we have ten hundreds.
So we must exchange that for one thousand bead.
And we write the 1 on the green line in the class of thousands.
And that is our last line on this form.

4. The child then adds all the missing zeros.
   Thus he sees how many places from one hierarchy to the next---the process of doing all the zeros at once.

   At 1000 we must write three zeros.
   We have moved three places to the left---three zeros---000.

EXERCISE #2: Reading and Writing of Numbers---and Forming

DIRECT AIM: To bring to the child's consciousness the function of the position; the relative value of the digits.

A. The teacher forms quantities and asks the child to read the quantity on the bead frame. Gradually increase the difficulty, including zeros in various positions.

B. The teacher writes a quantity on a slip of paper: 2,305. The child forms the quantity on the bead frame.

C. The child forms a quantity on the bead frame and then reads it. (reads it from the bottom.
Then, on the form, on the left side---he writes the quantity he has formed. (note that the comma must be written in; he will write the quantity units first)

Conclusion: To find out if the child has understood:

Form a simple quantity on one wire of the frame, and ask:

What is the value of this? (4 tens)
What is the value of these? (2 thousands)

OPERATIONS WITH THE BEAD FRAMES:
Addition: The Execution of Addition (concept of combining three quantities)

Presentation

1. The teacher writes the addition problem on a slip and the child copies it on the left side of the form: 2,324 + 3,567 + 1,458 =

2. The child first shows the first addent on the frame, forming in either direction.

3. Then the second addent is added to the beads, beginning with the units. When ten of one order is reached in this addition, the whole row must be passed back to the right side in exchange for one of the next order.
"I exchange these ten units then for one ten which I must now move over to the left."

4. The third addent is added in the same way.

5. The child reads the answer, and then writes it as the sum of his operation on the form.

Addition: To take the child to abstract addition

1. Proceed as in the writing of the operation above. . .and the first addent

2. The child then shows the units of the first addent, then combines the units of the second and third addents, exchanging for a ten when necessary. When all the units are shown, he notes the total units on the first wire and writes that number on the units line of his form as the first number of the sum.

3. Then he proceeds to the tens in the same way, writing that number of tens as a total when he has combined them all.

NOTE: This is a preparation for the abstract process of carrying over. When he proceeds to the next order, having written the total of the preceeding order, he finds that sometimes one of this new order has already been moved into the counted place at the left. It has been carried over.

A GAME: 999 + 1 =
A good game where we add only one unit, but on the frame this requires several changes.

Multiplication: with the multipliers 10, 100, 1000

1. Execute the multiplication 2 X 10 = on the bead frame, taking 2 units ten times. The child then reads the answer and writes it. Note that the first wire is empty and represents one zero; one place.

   1. We want to multiply 2 X 10. What does that mean? It means that we must take 2 units 10 times. So on the bead frame we first take 2 one time. . .2 two times. . .2 five times. I have used all the 10 units. I move them to the original place and show one ten. Now 2 taken six times. . .2 taken seven times. . .2 taken 10 times. And that gives me one more 10. Therefore 2 X 10 = 20. Notice that I have am empty place here. I have 2 tens and no units---I have to hold the units place with a zero. There are zero beads on this first wire.

2. 20 X 10 = Note the two empty wires signifying the two zeros holding the place.

   2. Now we want to take 20 ten times. We move two tens to the left. That is 20 taken one time. 20 taken two times. . 20 taken five times and I must exchange now these ten tens for one hundred, move the tens back and then 20 taken six times. . . So 20 X 10 = 200. This time I have two empty wires. What is holding those two places in our answer?

3. 3 X 100 = As in the work with the decimal material, show that one unit can be taken 100 times---and that is one hundred bead here.

   3. What happens if I want to take 3 a hundred times? I have to take the 3 units one hundred times. But if I take one unit a hundred times, it is one hundred. So we can show that with one red bead. If I take one more unit 100 times, that is 200. And if I take one more, that is another hundred. Our product, then is 300. How many empty wires do we have? We have two zeros which are holding the place.

Points of consciousness: the passage from one hierarchy to another is seen much more clearly here than in the work with the decimal materials. To point out that the passage from one hierarchy to another is always at 10.

Multiplication: One-Digit Multiplier

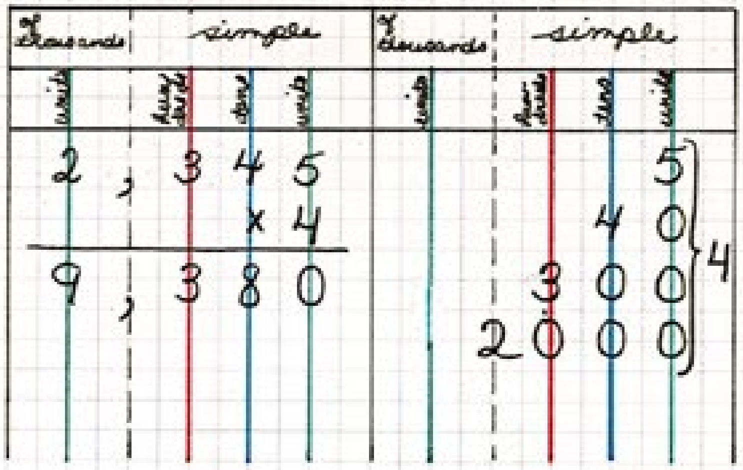
With the small bead frame, we only show multiplication with a one-digit multiplier; the larger multiplications must be done with the larger frame.

Presentation

1. The child no longer works with the form on which the numeration is shown on the left side. Now he writes the operation on the left side and on the right he shows an analysis of the multiplication.

THE BEAD FRAMES. . .
Multiplication:One-Digit Multiplier. . .
Presentation. . .

1. . .



2. With the bead frame, begin showing the multiplication with the units, taking 5 units four times. Proceed through 40 taken 4 times, 300 taken 4 times, 2000 taken 4 times. The child reads the product on the frame and writes the answer on the form on the left side.

2. We begin our multiplication by taking 5 units 4 times.
5 units taken one time. . .5 units taken two times. . .I must change now for one ten. . .5 units taken 3 times. . . . .
That is two tens.
Then 40 taken one time. . .

We can read our product on the frame: 9,380.

3. SECOND PASSAGE: The multiplication is done first from the analysis at the right side, and then those products are shown on the frame.

5 units X 4 is 20.
We move two tens to the left on the frame.
4 tens X 4 is 160 OR one ~~thousand~~ hundred and six ~~hundreds.~~ tens.
We move one ~~thousand~~ 100 bead to the left and six ~~hundred~~ 10 beads.
300 taken 4 times equals 1200 OR one thousand bead and two hundred beads.
2000 taken 4 times equals 8000---eight thousand beads.
Now we can read our product on the frame: 9,380, and write it on the form.

NOTE: This passage breaks the multiplication down and recalls the memorization of multiplication. . .but the child must also know the place of the digits, the relationship of zeros as a place holder for each hierarchy.
In the previous passage, we multiply the units first---as part of the movement to abstraction. . .but we may also

4. Multiply the various hierarchies at random.

Let's multiply first the hundreds X 4.
Then the units, then the tens, then the thousands.
We find that the answer is the same.
We can start from any place in our multiplication.
It is only important to show the quantity in the correct hierarchy for the relative value.

NOTE: This is possible only with the bead frame in which the place, the hierarchies are fixed.

DIRECT AIMS: to carry out addition, subtraction, and multiplication abstractly.

Subtraction

1. The child copies the operation onto the form at the left:

$$\begin{array}{r} 3{,}275 \\ -\ \underline{1{,}486} \end{array}$$

2. He shows the minuend on the bead frame.

3. Then, beginning with the units, he takes away the subtrahend in the bead quantities. When a greater number must be taken from a smaller one, he must borrow one units of the superior order in order to add ten to his order from which he must subtract.
   "I lose the ten. I move it to the right. I move ten units to the left."
   "Now I can subtract those I need."

4. The child reads the difference---the beads he has left on the left indicate this. And he writes the answer on the form.

Second Passage: the same process, but the child writes the total for the order as he makes it each time. Thus, he subtracts 6 from 5 units, makes the necessary exchange and then writes his answer: 9, on the form before proceeding to the tens.

A Preparation for Multiplication: A review of how to multiply quantities times 10, 100, 1000. The child's first intuition of this is given in the multiplication table of 10 and 1 in comparison. Later during the bead bar exercises when he multiplies a number by ten.

Presentation

1. Show the bar, square, cube, and bead. Make sure the child knows how are all composed in relation to each other.

   1. This is the 10. . .the 100. . .the 1000.
   How many bars of 10 are in this 100?
   How many 100-squares are in this 1000?
   How many units are in this bar?
   How many unit beads are there in the square?
   How many units are there in the cube?
   How many tens are there in the cube of 1000?

2. Prepare a multiplication on a slip: 123 X 10 =
   The child shows 123 with the decimal materials 10 times.
   He combines the materials, exchanging for the final product, which he reads: 1,230.
   Then he writes the answer in his notebook, showing the final zero of the answer in red. 123 X 10 = 1,230

3. The child shows with the decimal material: 20 X 10 =
   He shows two ten-bars 10 times, then combines the material, exchanging to get two hundred squares.
   And writes the answer: 20 X 10 = 200

4. Proceed to 4 X 10 = 40 which he may be able to do without the material.

5. Then 4 X 100 = .

   5. If we want to take this one unit one hundred times, what do we get?
   One hundred square.
   Then we will take it 3 more times and we get three more squares of 100.
   So I have 400.
   And we can write our answer with 2 red zeros: 400.

6. 25 X 100 = 2500. Here the child shows first the units times 100 and then the tens.

   6. One unit taken 100 times is one hundred square. And four more is five hundreds. One ten taken 100 times is one thousand cube. And another is 2 thousand.

THE BEAD FRAMES. . .
A Preparation for Multiplication. . .
Presentation. . .6. . .

6. So I have five hundreds and two thousands. . .
My answer is 2500.

7. 4 X 1000 = 4,000

7. If I take this unit 1000 times, I make one thousand cube.
Then I take the unit three more times.
My product is 4,000.
This time I have added three zeros to the multiplicand.

8. The child now has compiled a list:

123 X 10 = 1,230
20 X 10 = 200
4 X 10 = 40
4 X 100 = 400
25 X 100 = 2,500
4 X 1000 = 4,000

9. Give the rule.

9. Let's observe what we have.
Here the multiplicand was 123.
When we multiplied it by 10, all we did was add one zero.
Here the multiplicand was 4.
When we multiplied it by 10, we added one zero. . .here we added two zeros because our multiplier was 100.
How many zeros did we add when we multiplied it by 1000?

To our product we add as many zeros as we find in the multiplier.

The Real Material of the Hierarchies: A Geometrical Representation of the Decimal Hierarchy from Unit to Million: A Preparation for the Larger Bead Frame

Material: Geometrical figures made of light wood which represent the powers of ten. For an illustration of the second group of figures, the large bar, square and cube; see the powers of ten chapter. The first three figures resemble the golden bead equipment for unit, ten-bar and square in shape---and cube---but are considerably smaller, in exact proportion to the last three. Also, of course, the first four figures are squared whereas the bead material does not have the exactness of squared angles.

10^0 : a tiny green cube. . . . . 5mm. X 5mm. X 5mm.

10^1 : a very small blue bar with green lines to show that it has been formed by ten of the unit cubes. . . . . .5mm. X 5mm. X 5cm.

10^2 : a small red square with the ten bars of which it is formed marked in blue lines. . . . . 5cm. X 5cm. X 5mm.

10^3 : a green cube with red lines around the periphery indicated the 10 superimposed squares of which it is formed. . . . . 5cm. X 5cm. X 5cm.

10^4 : a blue bar with green lines indicating the ten cubes of which it is formed. . . . . . . 5cm. X 5cm. X 50cm.

10^5 : a large red square with blue lines indicating the ten blue bars of which it is formed. . . . . 50cm. X 50cm. X 5cm.

10^6 : a very large green cube with red lines indicating the ten big red squares superimposed which form the million cube. . . . . 50cm. X 50cm. X 50cm.

Presentation

1. Present and describe the materials listed above, using the decimal material for comparison of the first four. Consider the value of each figure.

1. The color of the units in the decimal system was gold because we gave our numeration a physical value. We didn't need different colors to show us which hierarchy we were using or working with. We could count all the beads.
Here the unit is this tiny green cube. The color gives the value.

The small blue bar corresponds to the ten-bar.
It is blue because it is a ten.

This square is red because it is a hundred.
It is made of ten blue bars; and it, in fact, corresponds to our golden square.

This cube represents 1000 because it is formed of 10 hundreds like our golden cube.

This big blue bar is formed of 10 cubes of 1000. Therefore, this is 10,000.
It is blue because it is a ten---a ten thousand.

THE BEAD FRAMES. . .

Presentation of the Real Hierarchical Material. . .

1. . description of the materials and their value.

1. . .This big square is formed of ten 10,000; and we call it 100,000. Its value is 100,000.
Remember each strip on the square is formed of 10 of these cubes.
So here we have 10 thousand, 20 thousand, 30 thusand. . .90 thousand. . .
100 thousand.
Because it is 100, it is red.

And here we have the large green cube.
If we take 100,000 ten times, we obtain one million.
This cube is 1,000,000.
It is green because it is the unit of the millions.

2. Display the three cubes together.

2. Here we have the cube---the simple unit.
Here is the unit of the thousands,
and here is the unit of the millions.

3. Compare the two red squares.

3. The small red square represents the hundred.
The larger one represents the hundreds of thousands.

4. Compare the blue bars.

4. There are two blue bars: the small one is the simple ten.
The larger one is the tens of thousands.

5. To see if the child has understood:

5. Bring me the material that corresponds to 1000.
Bring me the material that corresponds to 10,000.
Why is this the 10,000?

New Numerals

Materials: A set of numeral cards with the numerals in black and showing the commas in the correct place for: 1; 10; 100; 1,000; 10,000; 100,000; 1,000,000.

Presentation

1. Introduce the new numerals. Begin with the four he knows.

1. This is the numeral card 1---it corresponds to the simple unit.
This is 10, this is 100, this is 1000.

This is the numeral card for 10,000.
The material is, in fact, formed of ten thousands.
This is 100,000.
The square was made of 100,000 (a hundred thousands.)
This is 1,000,000. It has six zeros.
We have gone from the unit to the million.
The million is formed of a thousand thousands.

2. Second and third period lessons.

Matching the New Symbols to the Geometric Hierarchical Materials

1. Display the symbols on the materials. 1. This numeral corresponds to the unit.
   This one corresponds to the small bar.
   This numeral corresponds to the 100-square.
   etc.

2. Remove the numeral cards and ask the child to match the symbol with the quantity.
   a) Give the child the symbols and he matches with the material.
   b) Indicate a quantity and the child must choose the correct symbol.

3. In conclusion: Note the progression of point, line, square. . .repeated.

DIRECT AIM: To make the child conscious of the hierarchies of the decimal system beyond 1,000.

Age: 6 - 7

THE SECOND BEAD FRAME

Material

1. The second bead frame (see illustration at the beginning of The Bead Frames.) This frame has seven bead rows, with a larger space between the third and fourth wires; and a larger space between the sixth and seventh wires: each indicating the passage from one class to the next. On the left side are numbers indicating the value of each bead in that row. On the right side are the names of the classes: simple, thousands, millions; respectively in white, grey and black, indicating the increase of value. The bead rows are: 1)green for simple units, 2) blue for simple tens, 3)red for simple hundreds, 4)green for units of thousands, 5)blue for tens of thousands, 6)red for hundreds of thousands, 7)green for units of millions.

2. The larger printed form, which shows the bead frame repeated twice.

3. The hierarchical materials.

Presentation

1. Show the hierarchical materials, one at a time, and note which row of beads corresponds to that quantity. Begin with the tiny cube---the simple unit.

1. Do you remember the value of this cube?
   It is equal to one simple unit.
   And each bead on this first row of our large bead frame is equal to one of these tiny cubes---to one unit.

   Do you remember the value of this blue bar?
   Its value is 10.
   Therefore, each blue bead on this wire has a value of 10.

   Do you remember the value of this square?
   It is 100.
   And the same is true of each red bead in this row. Each is worth 100.

   What was the value of this green cube?
   Ea ch bead in this row has the value of 1000.

   Do you remember how many 1000s we used to form this bar?
   Then each bead here represents 10,000.
   And these beads are blue.

THE SECOND BEAD FRAME. . .
Presentation. . .

1. . . .a comparison of the hierarchical materials and the frame wires.

1. . The value of this big square was 100,000. But we can also represent 100,000 with one of these red beads.
Each bead on this wire has the value of 100,000.

Remember the value of the big green cube? It was equal to 1,000,000.
Each of the beads on this row has the value of 1,000,000.

2. Point out the numbers on the left hand side of the frame which tell the value of each bead on that row: review the value of each wire's bead.

Note that three orders form one class.

Note the larger spaces between the 3rd and 4th and the 6th and 7th wires; thus giving the million two commas.

2. On the left side of the frame we see written the number which tells us the value of each bead on that wire.

We also see that these three orders form a first group, named on the right side: simple units.
These three orders (wires) make up the thousands.
And this last row is the units of the millions.

The million is written with 2 commas which correspond to the two bigger spaces between wires on the frame.
Where are those spaces?
Between the units and the thousands; and between the thousands and the millions.

3. Note the increase of value as ten times between one hierarchy and the next---both on the frame rows and with the hierarchical materials.

3. Have you noticed that between each hierarchy on the bead frame there is an increase of 10.
The ten bead is worth ten times what the one unit bead is worth.
The same thing is true with this material---the next largest piece is proportionally always 10 times larger.

To see if the child has understood the relationship between the material & the frame:

4. Second period lesson: several exercises in variation: Begin with the hierarchy material OR start with a bead and ask the child to point out the corresponding quantity.

4. Which piece of our material corresponds to this bead?
Show me the bead which corresponds to this large red square.
What is its value?

5. Move the three green unit beads across and ask the child to point out the corresponding material and tell the value.
Emphasize the positional value.

5. What material corresponds to this green bead? this one? this one?
Each is a single green bead.
How do you know that each one has a different value?

When he has understood the value of each bead, go on to writing the quantities on the form. . .(he knows the numeration to 1000). . .proceeding on the form to show the numeration for each bead to 1,000,000.

6. The child moves the beads one at a time across the frame, one wire at a time, counting first 1 - 9 units, and writing 1 - 9 on the first line of the form. Then he proceeds to 1 - 9 tens, counting the beads and writing a number for each one he moves on the form. When he reaches 9 of an order, he must pass the whole row of beads back, taking one bead of the next hierarchy across and passing to a new line on the form.

di milioni	di migliaia			semplici		
unità	centinaia	decine	unità	centinaia	decine	unità
1,	0	0	0,	0	0	0
		3	0,	0	0	0
	4	5	0,	0	0	0
1,	2	3	4,	5	6	7
3,	0	4	4,	9	5	2
2,	9	0	0,	7	6	1

di milioni	di migliaia			semplici		
unità	centinaia	decine	unità	centinaia	decine	unità
1,	0	0	0,	0	0	0

THE SECOND BEAD FRAME. . .
Presentation. . .

NOTE: The ch... he proceeds with the counting and numeration throug... have 10 loose beads, and so he must c... at the same time he moves... and we ta... the 1...

7. The chi... a
Emphasi... e next.
indica... he zero.
to the... he hundreds---
...the tens and
...eds---gives
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...are four zeros.
...10,000 repre-
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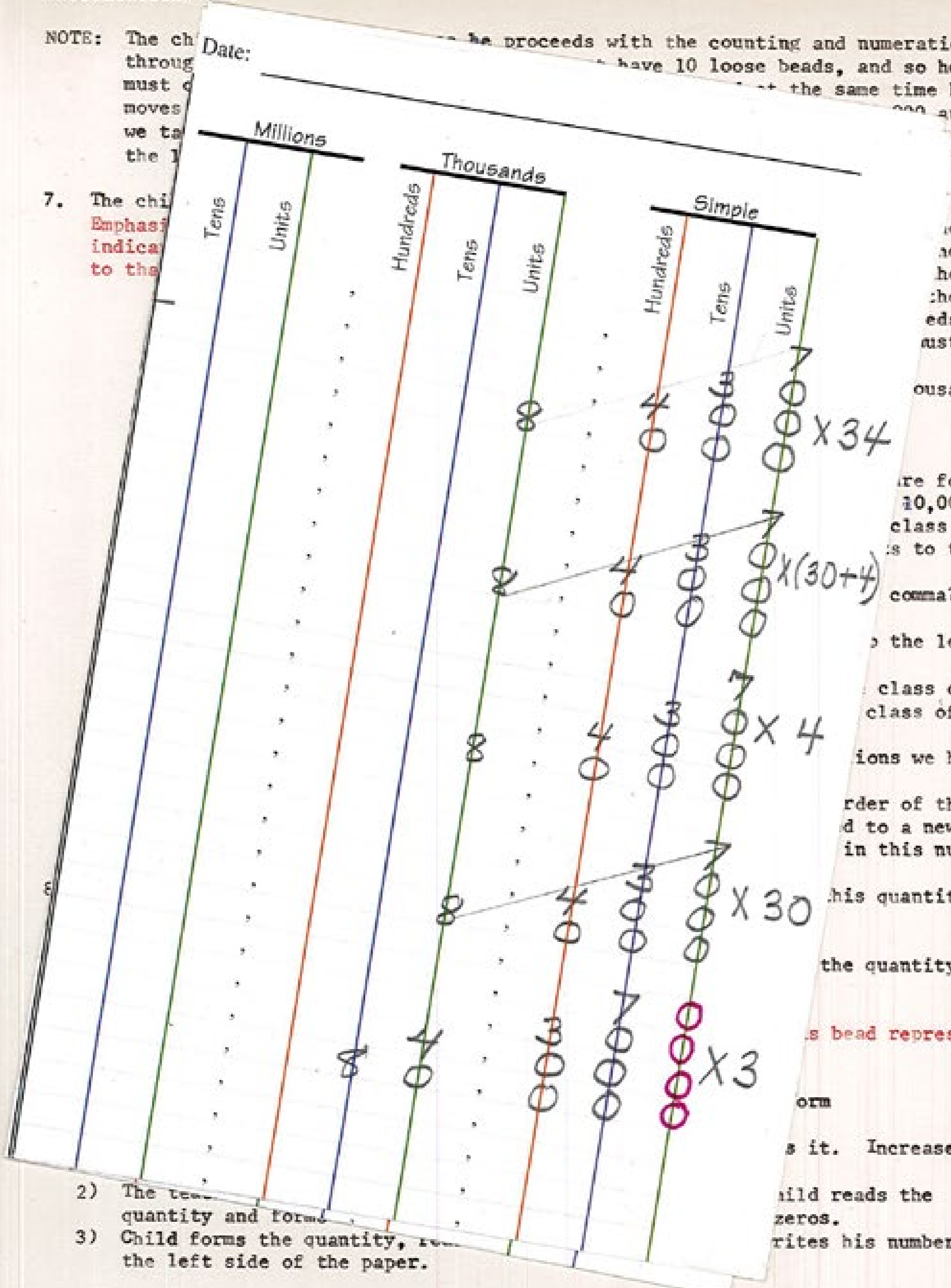


2) The tea... hild reads the quantity and form... zeros.
3) Child forms the quantity, r... rites his number on the left side of the paper.

THE OPERATIONS: Addition and Subtraction are presented as in the work with the first bead frame.

MULTIPLICATION: WITH A 2-DIGIT MULTIPLIER

Presentation: First Level

1. Write the multiplication operation on the left side of the form.
   1. 8,437 X 34 =

2. Review the decomposition of the multiplicand and show that decomposition on the right of the form.

3. Show a second decomposition in which both the multiplicand and the multiplier are broken down.
   3. We want to multiply 8,437 by 34. 34 is formed of 30 and 4. So we can write 30 + 4 times our multiplicand and we are also analyzing our multiplier.

4. Show the decomposed multiplicand now taken 4 times. . .that is, times the units.
   4. We first must multiply our number by the four units.

5. Cross out the first two decompositions and show the multiplicand times the tens.
   5. Then we want to multiply our number times the 30. We are not interested any longer in our first two decompositions, so let's cross them out.

6. Show the multiplicand as multiplied by ten in another decomposition and then multiplied by the simple units. The addition of the zero in this decomposition is written in red.
   6. How can we multiply this number times 30? We can multiply our multiplicand times 10. Now we still have four numbers, but the 7 has become 70; the 30 becomes 300; the 400 has become 4,000; and the 8,000 has become 80,000. Remember that to multiply a number times 10 it is enough to add a zero. Let's show that extra zero in red. How we can multiply all these numbers by 3.

7. Cross out the 4th decomposition, leaving only the two we will use.
   7. We can cross out our number times 30 now because instead we will use this decomposed number that we have multiplied by 10 times the 3 units.

8. Note that now there is simply the multiplications we have memorized.
   8. We can see now that we have only simple multiplication to do.

9. Multiply the quantity first times 4 on the bead frame. Do not write that product. Then show the product of the quantity times the 3. Write the total product, read from the frame, on the left side of the form.

Presentation: Second Passage: some time later

1. Do not do the first two decompositions. Begin with the multiplicand times the units. Then show the decomposition of the number times the tens; multiply the multiplicand times ten and show that decomposition times the units of the tens. Point out that we have divided the multiplier by 10.

2. Multiply the quantity times the units on the bead frame, read the product and write the partial product. Return the frame to zero. Multiply the quantity times the units of tens. Write the second partial product. The child adds for the total product. Left side of the form.

di milioni unità	di migliaia centinaia	di migliaia decine	di migliaia unità	semplici centinaia	semplici decine	semplici unità
			8,	4	3	7
		x			3	4
	2	8	6,	8	5	8
Second Passage:			8,	4	3	7
		x			3	4
		3	3,	7	4	8
	2	5	3,	1	1	0
	2	8	6,	8	5	8

di milioni unità	di migliaia centinaia	di migliaia decine	di migliaia unità	semplici centinaia	semplici decine	semplici unità	
						7	
					3	0	
				4	0	0	x 34
			8,	0	0	0	
						7	
					3	0	
				4	0	0	x(30+4)
			8,	0	0	0	
						7	
					3	0	
				4	0	0	x 4
			8,	0	0	0	
						7	
					3	0	
				4	0	0	x 30
			8,	0	0	0	
					7	0	
				3	0	0	
			4,	0	0	0	x 3
		8	0,	0	0	0	

MULTIPLICATION: WITH A 3-DIGIT MULTIPLIER (as shown on the form, next page)

DIRECT AIM: To take the child to the abstract level of multiplication through a work which requires an effort that forms and develops his mind. The bead frame work, in total, takes the child to the abstract level of addition, subtraction---and especially multiplication.

Presentation

1. The child writes the multiplication operation on the left side of the form.

2. Then he shows the multiplicand decomposed multiplied times the units and times the tens.

3. Now we can multiply the multiplicand times 10, adding a red zero to show that multiplication and our multiplier becomes the simple unit of the tens.

3. If we multiply the 6 units X 10, it becomes 60. . . We do not take into consideration the second decomposition now, so we can cancel it.

4. Show the multiplicand decomposed times the hundreds and then multiply that decomposed multiplicand times 100--- using two red zeros, which reduces the multiplier to the units of the 100s.

4. Now we can multiply our multiplicand times 100. How many zeros will we add to each part of the number? We'll write them both in red. And we can now show that quantity as multiplied simply by the units of the hundreds because our hundreds zeros are shown in the multiplicand. AND we can cancel the quantity decomposed X 300 because we have shown that part of the multiplication in a new way.

5. Note that only three of the decompositions are now necessary. And that the multiplication has been reduced to the simple multiplications.

5. Of our five decompositions we will consider only the first, the third and the last. In this way we have reduced a big multiplication to simple multiplications.

6. On the large bead frame, the child does the operation, first times the units, the first decomposition. He writes the partial product.

7. He returns the frame to zero and now multiplies the quantity of the multiplicand as shown in the third decomposition. He writes the partial product, including the zero for the units---there are no units on the first wire.

7. Now we return the frame to zero and multiply our multiplicand---which has been decomposed and multiplied by 10 here---times the units of the tens.

We know our product here---the partial product---will not start with the units because we have multiplied this number times 10---and we have no units. So there will be a zero in the units place.

8. Child returns the frame to zero, multiplies as in the fifth composition, and writes the partial product which will end now in two zeros.

8. Why does this partial product end in two zeros?

di milioni	di migliaia			semplici		
unità	centinaia	decine	unità	centinaia	decine	unità
			2,	8	3	6
				×3	2	4
		1	1,	3	4	4
		5	6,	7	2	0
	8	5	0,	8	0	0
	9	1	8,	8	6	4

di milioni	di migliaia			semplici			
unità	centinaia	decine	unità	centinaia	decine	unità	
						6	
					3	0	
				8	0	0	x 4
			2,	0	0	0	
						~~6~~	
					~~3~~	~~0~~	
				~~8~~	~~0~~	~~0~~	x 20
			~~2,~~	~~0~~	~~0~~	~~0~~	
					6	0	
				3	0	0	x 2
			8	0	0	0	
		2	0,	0	0	0	
						~~6~~	
					~~3~~	~~0~~	
				~~8~~	~~0~~	~~0~~	x 300
			~~2,~~	~~0~~	~~0~~	~~0~~	
				6	0	0	
			3,	0	0	0	
		8	0,	0	0	0	x 3
	2	0	0,	0	0	0	

9. The child does the addition of the three partial products abstractly.

NOTE: With this material, the position of the zeros in the partial products is very clearly demonstrated. When we multiply the multiplicand times units, the first digit will be units. When we multiply it times tens, the first digit will be in the place of the tens. . .and so forth.

When we multiply units X units, we get units.
When we multiply tens X units, we get tens.

It is important to prepare operations at first which do not give zeros in the place of the units, the tens, etc. . .so that he sees that the zeros are a result of the original multiplication of the multiplicand---and a reference to the multiplier.

A special case: 2,836 X 304. Here the child write a decomposition of the multiplicand times zero. This results in a row of zeros as a partid product. Gradually he realizes that he can skip this row altogether---that as soon as we write X 0 the whole number disappears. IT IS IMPORTANT THEN, THAT HE SEE THAT IN THE NEXT ROW, THERE WILL BE AN EXTRA ZERO IN ADDITION TO THE USUAL ADDITION OF ONE PLACE ZERO TO THE LEFT.

```
  2,836          2,836
  X 304        X   304
 11,344         11,344
 00,000        850,800
850,800        862,144
862,144
```

EXERCISE: to find out how a calculator works.
The children may discover the same principles of the zero. That is, that the multiplication times units gives units. BUT a multiplication times 10 gives a zero before the number; with 100 there are two. So the calculator, too does the position shifting.

```
321 X 24 =     321        4,782 X 324 =       4,782
               321                            4,782
               321                            4,782
               321                            4,782
             3.210                           47,820
             3.210                           47,820
             7.704                          478,200
                                            478,200
                                            478,200
                                          1,549,368
```

NOTES from Mario Montessori: The bead frames are the base for the understanding of the function of addition, subtraction, and especially multiplication with two or more digits in the multiplier. After these experiences, the child will be able to carry out the operations abstractly.

from Maria Montessori: Children do not condemn themselves with narrow limits. When the children are free to act, they choose the maximum effort. Repetition is true not only for practical exercises, but for those mental. Small children do not abandon a problem until they are the masters of it. The elementary child has the same need for repetition of his mental work. And so the children prefer long, big, week-long multiplications. The child looks for obstacles to overcome. . .and this overcoming develops his mind.

AGE: 7 - 8

NOTE: The chief aim of the bead frames is to emphasize the movement connected with zeros. The Montessori solution to this understanding of hierarchies and the work done with them was to isolate the powers of 10 and transfer them from the multiplier to the multiplicand.

2,327 × 3,598 =

```
    7 }      8 → 8 · 10^0
   20 }  x  90 → 9 · 10^1
  300 }    300 → 3 · 10^2   then   multiplying by 10^N
2,000 }  2,000 → 3 · 10^3          and dividing by 10^N
```

THE BANK GAME: A parallel work with the bead frames.

This bank game gives the child a complete vision of the different aspects which constitute the process of carrying out a multiplication. It usually involves three children: one who does the multiplication, one who acts as the cashier and one who writes down what happens. In addition, there is usally a considerable audience.

Material

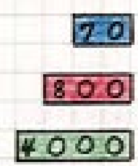
1. A wooden box with sufficient divisions to contain neatly all the following series of numeral cards.
2. 9 series of numeral cards, white with the orders in the hierarchical colors. The cards graduate in size according to the number of digits. . .that is, 1,000 card is four times larger than the 1.
   a) 1 - 9 Green.
   b) 10 - 90 Blue.
   c) 100 - 900 Red.
   d) 1,000 - 9,000 Green.
   e) 10,000 - 90,000 Blue.
   f) 100,000 - 900,000 Red.
   g) 1,000,000 - 9.000,000 Green.
   h) 10,000,000 - 90,000,000 Blue.
   i) 100,000,000 - 900,000,000 Red.
   All these numeral cards have the function of the product.
3. 4 series of numeral cards, these cards in the hierarchical colors with black numerals. They represent the multiplicand.
   a) 1 - 9 Green.
   b) 10 - 90 Blue.
   c) 100 - 900 Red.
   d) 1,000 - 9,000 Green.
4. 3 series of numeral cards, each series going from 1 - 9; the cards are grey with black numerals.
   In addition to this series, there are 3 cards, grey with black numerals (zeros), one card with 0, one with 00, one with 000.
   This series represents the multiplier.

THE GAME: Presentation in Preparation for The Game: Acquaintance with Materials

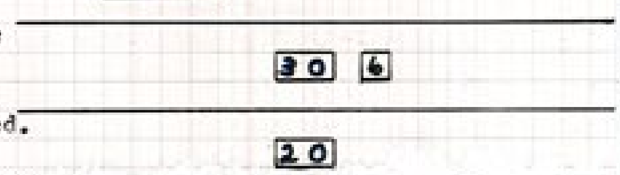
1. Write the multiplication on a slip: 4,876 X 6 =

2. The multiplicand is formed with the colored numerals, the child plays the magic game to form the whole number. Then the multiplier is shown with the grey cards and slips are prepared for the signs of multiplication.

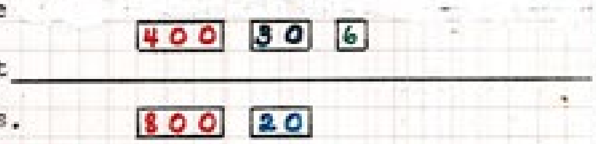
3. Decompose the multiplicand.



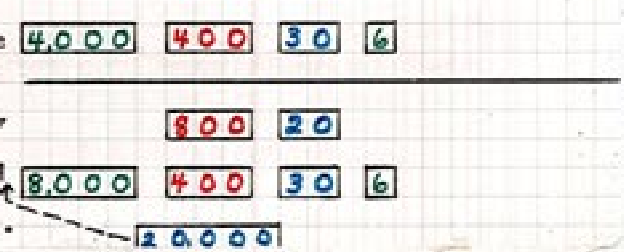
3. The child multiplies first the units times the multiplier and the cashier shows the product at a place below on the mat with the white numeral cards---in combination---but not united.



4. Then the multiplier and multiplication signs are moved into position beside the tens and that multiplication is made and shown with the first partial product as a group of white cards---it is important that the orders be kept in rows.



5. Then the multiplier and signs are moved to the 100s and the 1000s, each multiplication being added to the group of product cards.
IF a product numeral is needed that is already shown in the answer, the cashier must either use a combination OR---he must change the card shown so that he represents both numbers with one larger card. Two 4000s---he uses the 8000.



6. The child #1, the multiplying child, now exchanges cards to show only one numeral card for each hierarchy. The cashier makes the exchanges.

7. From those final product cards, the magic game produces the whole number and the multiplicand is again shown whole. The result is displayed on the mat and the child #3, the scribe, writes it all down.

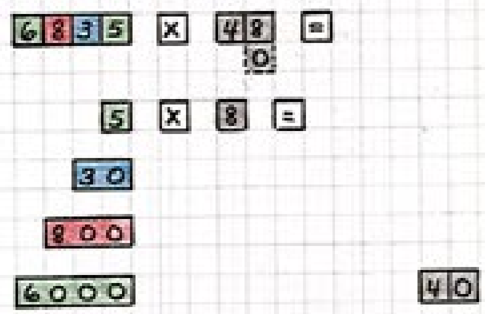
THE BANK GAME: Two - Digit Multiplier

1. The child shows the multiplication: the multiplicand is composed of the colored numeral cards. The multiplier now has two digits and so beneath the unit numeral card, which is gray, we place the zero digit card indicating the place-holder of the tens.

1. Because our multiplier is two-digits now, we form it with the two grey digits cards, but we also must have a zero card below the units. That is the zero of our tens.

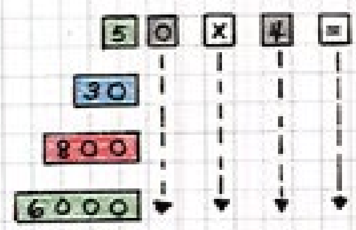
2. The multiplicand is shown, as before, decomposed. But now, we must also show the multiplier decomposed.

3. The multiplication begins times the simple units, proceeding as in the first presentation; the child making the necessary exchanges with the cashier to show his products.



4. Then the children will discover that at this point the product should be simplified to one numeral card in each hierarchy to simplify the following work.

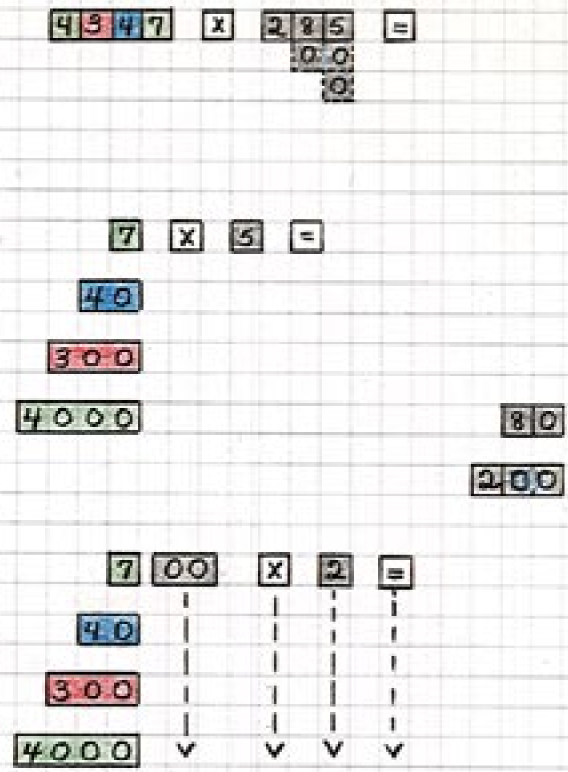
5. Now the decomposed multiplicand is multiplied times the tens. In order to do this the child takes the zero, as he did on the form, from the ten and places it before each decomposed part of the multiplicand---thus multiplying it by ten---and his multiplication is reduced to the simple multiplications times the units of the tens. So he will have not 5, but 50 times 4 (not 40). Note that when we multiply the multiplicand times 10, we are dividing the multiplier by 10. WHEN WE MOVE THE MULTIPLIER DOWN TO THE SECOND PART OF THE DECOMPOSED MULTIPLICAND, WE ALSO MOVE THE ZERO DOWN.



6. Now each hierarchy is again reduced to one card---many changes necessary here. The total product is formed with the magic game and the complete multiplication shown.

CONTROL: the teacher OR booklets may be prepared with multiplications and an accompanying booklet of products which the children may consult at the end.

Three-Digit Multiplication



1. The multiplication is shown as before, but now there are two zeros below the units (the place holders for the tens and the hundreds), and one zero below the tens (place holder for the hundreds).

2. The multiplicand is decomposed and also the multiplier.

3. The multiplication then begins, as in the previous games, with the multiplicand times the simple units. Then, showing the zero before the decomposed multiplicand the multiplication is made times the units of the tens as in the two-digit multiplication.

4. Finally, the decomposed multiplicand is multiplied times 100 (shown with the two zeros of the multiplier before each of the parts) and then multiplied times the units of the hundreds.

5. The numeral cards are simplified after each multiplication of one order; but do not tell the children to do this; they will discover it as the easiest way.

6. After the final multiplication, the hierarchies are again reduced to one card each, the whole number is made with the magic game and the multiplication shown. The scribe records.

7. The children CONTROL with a prepared booklet of products or the teacher checks the answer.

DIRECT AIM OF THE BANK GAME: To acquire ability and speed in making the changes necessary for the compilation of the final product.

AGE: 7, 8

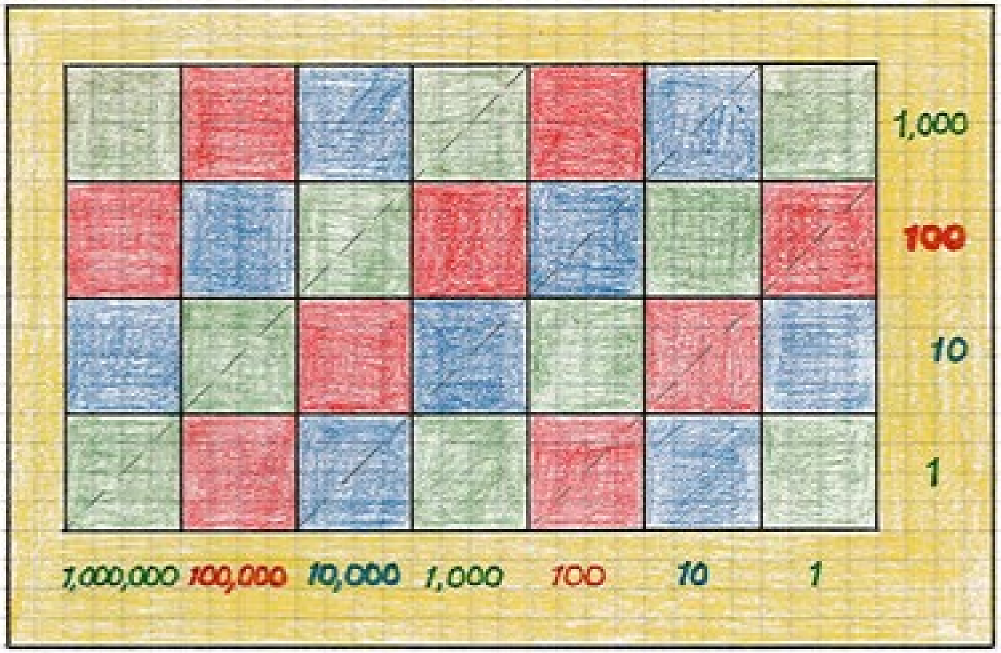
NOTE: from Maria Montessori: Material for development is that which determines an active work which develops the child's potentialities. Here, the development of the mathematical mind, which is one of the attributes and needs of the human mind. That mind is always reaching towards the development of its potentialities: but if they are not stimulated and allowed to exercise, the development of the potentialities does not occur.

NOTE: from Signore Grazzini: In the Bank Game we still have the analysis of multiplication: but the work of the hierarchy positions is now done with symbols. We see again how the zero in its position determines the value of the digit. And we see that when we multiply the multiplicand by a 10 or a 100 we are, at the same time, dividing the multiplier by 10, 100, etc.

The Aim of the Bank Game is parallel with that of the Bead Frames: to introduce the place value, the relative value of the digit. To emphasize the movement connected with the zero.

THE CHECKER BOARD : Parallel with the Bank Game and the Bead Frames: The Aim's the Same

DIRECT AIM:



Materials

1. The checker board as shown, constructed of wood and decked with patches (squares of felt). The side of each square is 7 cm., to contain a 9 bead-bar diagonally across the square.
   NOTE: the child can locate the value of the internal squares on the board by following the diagonals to either side of numerals.

2. The box of bead bars 1 - 9: NO TENS. Cover them if used.

3. 3 series of small numeral cards: white with black numerals for the multiplicand. Each series has 1 - 9.

4. 3 series of small numeral cards: 1 - 9, grey cards with black numerals. Represents the multiplier.

Presentation: Familiarizing with the Materials

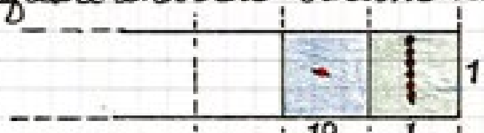
1. Introduce the child to the checker board, noting the numerals on the two sides and the hierarchical colors of the squares. Note with him especially the value of the internal squares, found by following the diagonals. Be sure he knows the value of each square. Introduce the numeral cards later, when the operation is presented.

2. Then place a three-bar on the bottom row, first 10 square. He identifies the value.
   . . . then on the unit square.
   . . . then on the 10 square above the unit.

2. What is the value of this bar if I place it here?
   30
   What is the value now?
   3
   What is the value here?
   30

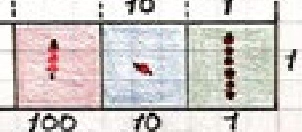
3. Now use 2 bars, one on the unit square and the bottom row 10...

3. What is the value of this quantity?



16

then 3 bars...

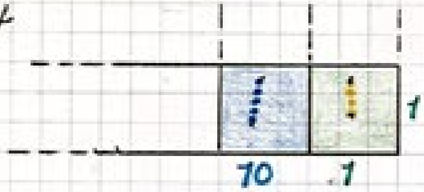


If we add a bar, what is the value here?
316

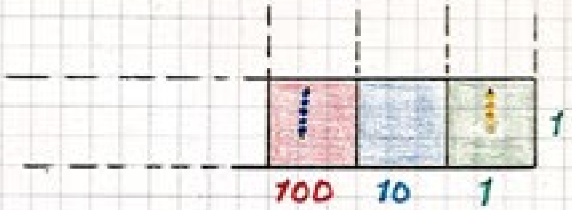
The Checker Board . . .

Exercises: to make sure the child understands the value of every square . . . that he can find the value on the diagonal . . . and that he understands the hierarchical movement.

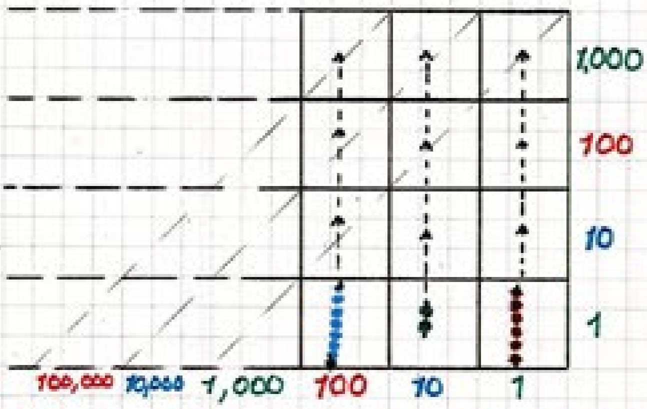
4. Using 2 bars, now show
first 54
then 504



What is the value?



5. Using a 3-digit number, place the bead bars on the bottom row. The child reads the quantity. Then move those three bars up to the second row. The child reads the new quantity. Continue to the third and fourth rows.



= 826,000
= 82,600
= 8,260
= 826

Every time we move the bars one row up, we are multiplying by ten powers of 10 and going up one hierarchy.

6. Repeat the exercise, with two bars.

7. Then move those bars down showing: and that here we are dividing each time by 10 and going to a lower hierarchy.

23,000
2,300
230
23

8. Using a 4-digit quantity, show the bars in several different positions. The child reads the quantity.

D = 8,675
C = 8,675
B = 8,675
A = 8,675

4234 (1) = 4,234

4 (10)
423 (1) = 463

4 (100)
3 (10)
42 (1) = 472

4 (1000)
3 (100)
2 (10)
4 (1) = 4,324

1000 100 10 1

Note: With the checker board, we are still in multiplication, but it is multiplication from a geometric point of view. 3×4 gives us a rectangle that we will be able to visualize on this board.

The Checker Board . . .

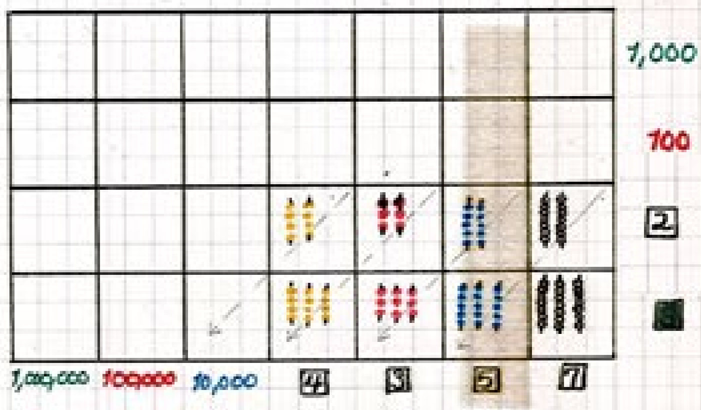
2 - Digit Multiplication : First Passage : showing the products as a combination of the number of bead bars of each quantity in the multiplicand taken x number of times.

1. Write the multiplication on a slip : 4357 X 23.

2. Introduce the three series of white cards with which the child forms the multiplicand at the bottom of the board, placing the cards over the numerals; and the three grey series with which he forms the multiplier along the right side over the numerals.

3. Then the child first makes the multiplication times the units, showing the product as a combination of bead bars on the square.
   7 taken 2 times = three seven bars

4. When he completes the unit multiplication, he turns over the unit card and proceeds with the multiplication of the tens.
   7 taken 2 times = two seven bars on the ten square.



5. When the bars are shown as products thusly, the child slides the bars above the bottom row down to it . . . bringing on the diagonal all the 10s to the 10 square, all the 100s to the bottom hundred square, etc.

6. The child exchanges the bars now so that he shows only one bar for the units, one for the hundreds and so on. This involves the process of carrying over.
   So 3 7-bars = 21. We exchange for one unit bead which we show in the unit square. AND two tens which we show as the two-bar and carry it over to the ten square.

Completed, our answer looks like this — shown on the bottom row:

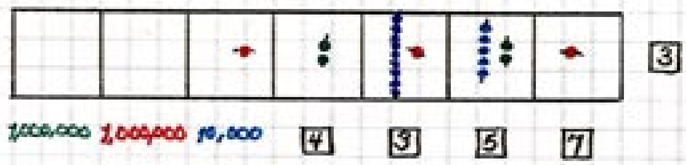
4,357 X 23 = 10,0211

The Checker Board

3- Digit Multiplication : Product simplified to one bead bar for each square as the simple multiplications are made.

After the child has worked for awhile with the first passage, he does this work: showing for each simple multiplication only one bar in each hierarchy square. Therefore, 7 x 3 = one unit bead on the unit square and the two - bar on the ten square.

Then, in the multiplication 4, 357 x 423, the bottom row looks like this.



1. The child shows his simple multiplications as here shown, beginning with the multiplication of the units, then times the tens, then hundreds.
2. Then he slides all bars of the same hierarchy along the diagonal to the bottom row.
3. He combines those bars now found in the bottom row squares to represent each hierarchy with one bar. Again he is carrying over.

Note : the child is reinforcing the idea that when we multiply units x units our first digit is units ; units x tens means our first digit is tens, etc.

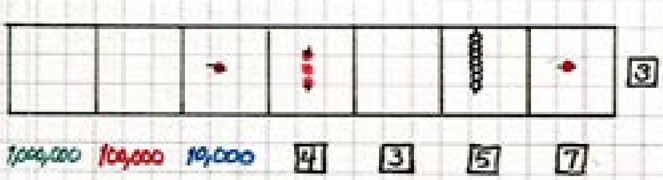
Considering the Partial Products: 3rd Passage

Here the child simplifies each completed row of multiplication on the board, reducing each hierarchy to one bead before he passes on to the next row of multiplication.

He may or may not write each partial product. If he does, he must remember that in the second row, the final square is tens which determines the position of his partial product in his written operation. NOW we drop the extra zero in the partial product, telling the child that the value remains the same without that zero if we get the digits in the right columns.

So the above first row combined as a partial product becomes . . . . and is written

```
   4,357
    x423
  13,071
  14714
 17428
```



And the position of the next partial product must be carefully noted.

The child slides the bead-bars down the diagonals, combines the bars on the bottom row to simplify the answer to one bead bar per square, reads the answer and writes it

The Checker Board . . .

Mental Carry-Over: 4th Passage

As the child makes each simple multiplication, he places only one bead-bar on the square above his multiplicand digit and remembers the number carried over to the next hierarchy.

So 7 × 3 = one unit bead on the unit square and two tens to keep in mind for the next operation

then 5(0) × 3 = fifteen or the five plus the two carried over — shown as a seven bar and one ten to keep in mind for the next operation.

And so, at the end of the row it is not necessary to combine for the partial product. That partial product is already shown.

Note: emphasize throughout that we may not combine the quantities vertically or horizontally. That by combining on the diagonal, we are putting all the beads of the same hierarchy together.

ACTIVITY: Multiplication with the Checkerboard and the Design

This activity is limited to the first passage.

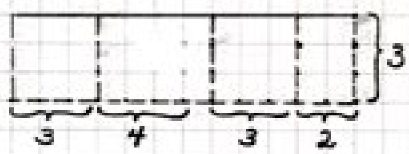
1. Write the multiplication. The child lays out the white multiplicand numerals at the bottom of the board and the grey numerals along the right side to show the multiplier.

1. I want to multiply 3,432 X 43 =

2. The child proceeds with the multiplication of the multiplicand times the unit 3. He shows each product on the squares as a combination of the same quantity bead bars.

2. Now, as we did in our very first game with the checkerboard, we multiply 3 X 2. We are taking the units 2 three times, so we show that with three two-bars.
Then I take the 3-bar three times on the second square---the blue square.
And the 4-bar three times on the third square---the red square.
And the 3-bar again three times on the next square, the fourth---the green square.

3. At the end of this first row of multiplication, the activity begins. The child, working on squared paper marks off squares and rectangles showing each of the small multiplications.



He then joins the points of each figure and he has formed rectangles and squares.

3. We stop here and begin our design.
We start with the units.
On our paper we mark off two squares because the base of the multiplication here on the first green square is 2.
We have taken that 2 three times.
Our second base line will be 3 and we have again taken that three times, so our line going up is again 3.

Now we can draw lines between these points to form geometric figures.
Which ones do you recognize?

4. The child now colors the rectangles. . . he reasons the color through a written calculation of what his multiplication is for that square. . . he writes the simple product on the square OR he may prefer to write all the numbers as units.

4. We color the first rectangle green.
Why?
Because units times units gives units.
How do we obtain this rectangle?
By multiplying 2 X 3, and that gives us six units.
We write 6 on the rectangle.
The second rectangle is blue because we have multiplied units X tens and that gives tens.
How did we get this rectangle?
By multiplying 3 X 30 (three tens).
Our result is 9 tens or 90 units.

First series calculations: bottom row

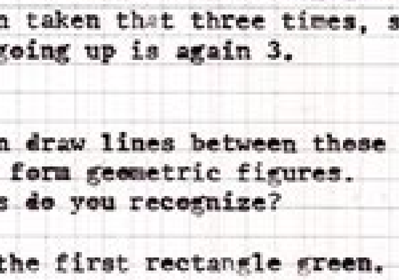
u x u = u   2x3 = 6
t x u = t   3x3 = 90
h x u = h   4x3 = 1200
k x u = k   3x3 = 9000

10,296

Second series calculations: second row

t x u = t   2x4 = 80
t x t = h   3x4 = 1200
h x t = k   4x4 = 16,000
k x t = M   3x4 = 120,000

13728
147,576



Note: the child may prefer to write the numbers instead of the letter symbols:
so u x u = u is 1 x 1 = 1
and h x t = k is 100 x 10 = 1,000

5. The child combines the bead bars shown on the first row for his first partial product. Then he checks that product with the one he has calculated from the design.

6. He proceeds with the multiplication by the tens. . .as in the first passage multiplication. The two-bar taken 4 times, etc. . .

7. He expands his design to a second row of rectangles and squares, based on those multiplications. He makes the corresponding calculations.

7. In our first multiplication that we want to show in our design, the base remains the same---2. But the rectangle changes because we have multiplied by 4 (tens).
So the rectangle will be 4 squares tall.

Why is our first rectangle blue?
We have multiplied units times tens.
And our result is tens. How many?
Why is the second one red?
Because I have multiplied tens times tens

8. The child combines the bars of the second row for the second partial product and checks it with his design calculation.

9. Now he combines the beads to find the total; and he combines the written partial products to check that total. NOTE: it is important that he correctly place the second partial product below the first with the first digit of that second partial product in the tens column.

10. Now he writes below and to the right side of his design the numbers representing the multiplication.

11. Observe with the child the geometric figures formed.

11. This rectangle represents the multiplication 3,432 X 34.
It is formed by squares and rectangles.
Where are the squares?

NOTE: If the child multiplies 4 digits by 4 digits, he creates a larger design with more squares and rectangles.
NOTE: On materials: whenever this checkerboard game is presented, the numeral cards are first laid out in order to the side.

DIRECT AIM: To point out how the products are obtained.

12. Make a final observation of the checkerboard with the child, considering the diagonals and the digits on two sides which identify that square as a product of two numbers, two hierarchies.

THIS IS IMPORTANT WORK: the child might make a diagram of the board, showing the combinations of the powers which result in the square's value.

12. Let's observe our checkerboard.
We have one unit square: it shows us units times units: 1 X 1 = 1.
But there are two squares for the tens: units times tens and tens times units:
1 X 10 and 10 X 1.
How many possibilities are there for the hundreds?
We have three hundred squares:
100 X 1, 10 X 10, and 1 X 100.
How many thousand squares are there?
What products do they show?
1,000 X 1 = 1,000
100 X 10 = 1,000
10 X 100 = 1,000
1 X 1,000 = 1,000

THE CHECKERBOARD. . .
The Design Activity. . .

1,000,000 x 1,000 = 1,000,000,000	100,000 x 1,000 = 100,000,000	10,000 x 1,000 = 10,000,000	1,000 x 1,000 = 1,000,000	100 x 1,000 = 100,000	10 x 1,000 = 10,000	1 x 1,000 = 1,000	1,000
1,000,000 x 100 = 100,000,000	100,000 x 100 = 10,000,000	10,000 x 100 = 1,000,000	1,000 x 100 = 100,000	100 x 100 = 10,000	10 x 100 = 1,000	1 x 100 = 100	100
1,000,000 x 10 = 10,000,000	100,000 x 10 = 1,000,000	10,000 x 10 = 100,000	1,000 x 10 = 10,000	100 x 10 = 1,000	10 x 10 = 100	1 x 10 = 10	10
1,000,000 x 1 = 1,000,000	100,000 x 1 = 100,000	10,000 x 1 = 10,000	1,000 x 1 = 1,000	100 x 1 = 100	10 x 1 = 10	1 x 1 = 1	1
1,000,000	100,000	10,000	1,000	100	10	1	

We can understand now why we always move the products down on the diagonal. We bring these beads of the same hierarchies, the same value, together. The diagonal is a result of the organization of the hierarchy.

THE DESIGN ACTIVITY: A final passage: To Abstraction

Here the child is writing multiplication in abstract form:

```
 3,432
  X 43
```

AND solving it without the materials. However, often the children continue to write down how the products are formed, showing each product in the multiplication. And they continue, too, to make the designs <u>without</u> the materials.

FINAL NOTE: The children like this material because it presents a new difficulty in multiplication. When they have mounted that particular difficulty, they abandon the game. The material demands a mental effort, which pleases the child.

DIRECT AIM: To review the concept of hierarchy and the geometrical presentation of multiplication.

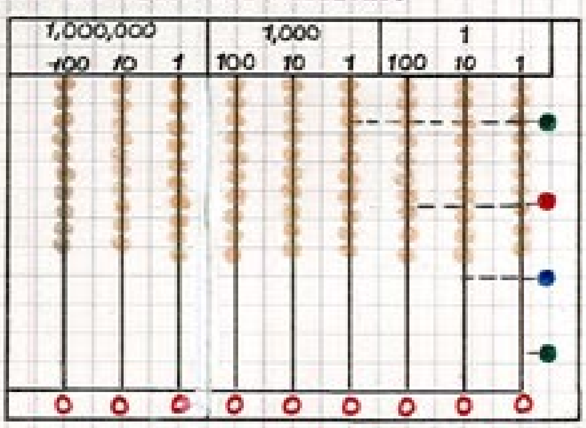
INDIRECT AIM: A remote preparation for the square root and the perfect squares because, with a real understanding of the hierarchies and how they are formed, it is possible to understand the square root.

THE GOLDEN BEAD FRAME: Parallel in aim to the bead frames, the checkerboard and the bank game.

This game presents a new difficulty even though it is still multiplication. It presents one more step towards abstraction. The sensorial stimulus of the hierarchical colors is gone. There remains only the three colors of the classes at the top: white for units, grey for thousands and black for millions.

Material

1. The golden bead frame.



2. A series of grey numeral cards with black numerals: 1 - 9 in two sets to represent the multiplier.
3. Slips of paper.

Presentation: First Passage: Without the partial products

1. Show the bead frame with all of the beads positioned at the top. The child writes the multiplication: 6,542
   X 36

2. He shows the multiplier with the grey numeral cards, placing the unit (6) card over the lower green dot (units) and the ten (3) card over the second dot which is blue for tens.

3. Then he folds a long strip of paper over the bottom of the frame, extending from the units wire to the units of thousands since his multiplicand is a four-digit number. Then, below each wire, he writes the numeral of the multiplicand.

4. First he makes his multiplication times the units. He multiplies 2 times 6 which means that he must bring down 2 unit beads on the first wire and 1 bead on the ten wire. He proceeds: 4 taken 6 times:= four tens taken six times, so that is 24 tens or 240 which is 4 beads down on the ten wire and 2 beads down on the hundred wire.
   ----5 X 6 = 5 hundreds times six, so that is 30 hundreds or 3,000 and that is 3 beads down on the fourth wire--- the units of thousands.
   ----6 X 6 = 6 thousands times six or 36 thousands which is 3 beads down on the tens of thousands wire and 6 beads down on the thousands wire.

5. **When the first multiplication series is completed, the child turns the unit multiplier face down and proceeds with the multiplication by tens. HERE HIS FIRST PRODUCT WILL BE TENS SO THERE WILL BE NO MOVEMENT ON THE FIRST WIRE. He completes the multiplication and makes the necessary exchanges for one of the higher order when his number of beads in an order goes to ten.
   **MOST IMPORTANT in this multiplication is that the multiplication by tens is achieved by moving the bottom slip of paper over to the left one place, and thus revealing one red zero. In other words, we have again multiplied the multiplicand by the power of 10 so that we can continue with our simple multiplications times a unit multiplier.
   NOTE: Another guide to this multiplication is the line drawn from the blue dot to the second wire, thus indicating that when the multiplication is times the digit shown on the blue dot, the first product will begin on the ten wire.

6. The child reads his answer after this second multiplication is completed. And he writes the total.

NOTE: In this game, we are using the nomenclature of the hierarchy, using the top labels for reference. That is, we are saying "six hundreds taken three times," and we can see that it is six hundreds by looking to the top of the wire.

Second Passage: With partial products: Three-Digit Multiplication

The game is the same with the exception that, after the first multiplication series times the units, the partial product is read and written and the beads are returned to the top of the frame to begin the second multiplication times tens. When this second partial product is completed and read, it should be noted again that, just as the first quantity of the product on the frame was on the tens wire, so we must write that first number of the partial product as a ten.

Proceeding to the multiplication series by the hundreds, the paper slip at the bottom is again moved one place to the left so that the multiplicand has been multiplied by 100 and two red zeros are shown.

Third Passage: Mental Carry-Over

In this game, the child only moves the beads on one wire for each multiplication. He then remembers the quantity remaining for the next wire, makes the next multiplication and adds that quantity to the first digit of his new product. Then shows that on the wire. So when 5 is multiplied by 7, the child first has a product of 35, but he brings down only 5 beads on the unit wire and remembers the three tens. If the next multiplication is 2 times 7, then he has four tens plus the three he has in mind and he moves 7 beads down on the ten wire, remembering that he has one hundred to carry over.

Conclusion: With the golden bead frame, we have moved one more step towards abstraction beyond the second bead frame. With both the frames, the digits of the multiplier, if we divide them by 10 or the powers of 10, become units. And the multiplication becomes larger by the power of 10 by which it has been multiplied. In this passage of multiplication and division by the powers of 10 with the second bead frame, it is only after decomposing the multiplicand that one, two, or three zeros were added. On this golden frame this operation is done by moving the whole multiplicand one, two, or three places to the left, thus reducing our multiplier to simple units. So the method is the same, but the process is different.

Thus we understand why, when we multiply by several digits, the first digit of every partial product must be written below the PLACE OF THE MULTIPLIER DIGIT by which we are multiplying.

AGE: 9

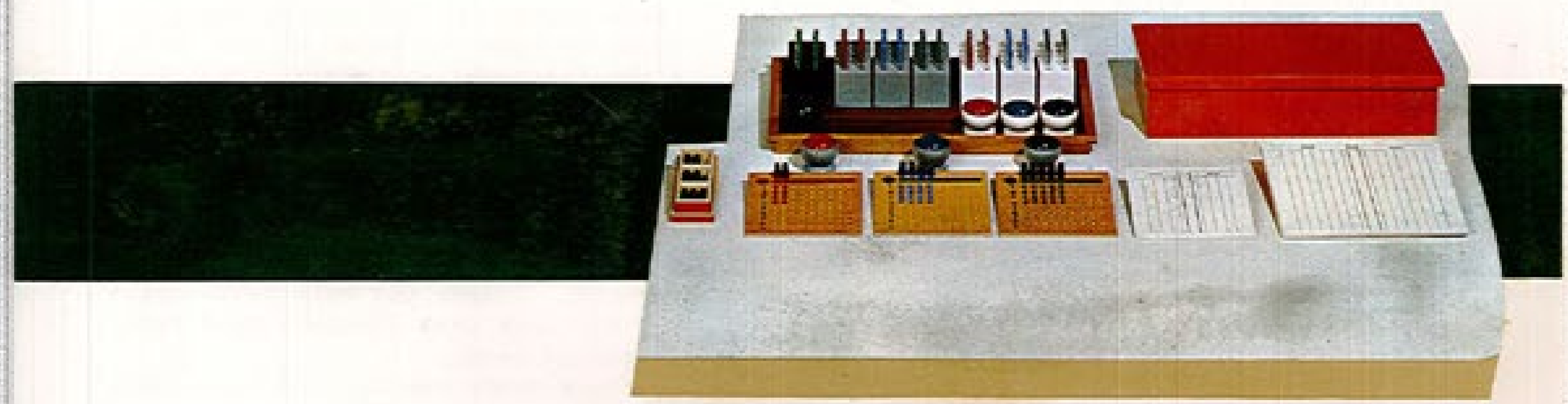
NOTE: Parallel in aim with the second bead frame are the Bank Game, the Checkerboard, and the Golden Bead Frame. It is interesting to carry out a long multiplication with all four materials. By different roads, the materials bring to the child's consciousness the different positions of the digits of a number when multiplied by big multipliers.

DIVISION WITH THE HIERARCHICAL MATERIAL

The work in division began with the decimal system material. At that time the child was introduced to one-digit divisors, to decurion division and centurion division. His first division work with the hierarchical materials came with the stamp game. Then we proceeded to the memorization of division. Here we were limited; there were some possibilities that could not be examined. Much importance was given here to the necessary dividends. With the bead frames, we were not able to do division because the beads were not loose.

Now we proceed to material with which we can examine all cases of division. With this material we will emphasize what one unit receives; not only as a quotient, but as the value that each digit of the quotient has. We can carry out division with two and three-digit divisors. Dott.sa Montessori says that this material can become a pastime because of the nature of the material itself. It takes the child to a constant analysis and to rational arithmetic.

The Material



1. 9 small dishes which hold the beads of hierarchical colors representing the dividend. Recalling the hierarchical colors, there are three dishes white on the outside to hold the three orders or units and the colors inside are green, blue and red. There are three dishes grey on the outside representing the class of thousands, and again the three are inside colored green, blue and red. And finally, there is one black dish, colored green inside. The BEADS WILL TAKE THE VALUE ACCORDING TO THE BOWL THEY ARE IN.
2. Corresponding to each of the dishes there is a rectangular test-tube holder, three white, three grey, one black. Each of the holders has places for ten test-tubes.
3. There are, therefore, ten test-tubes which fit into each of the seven holders. And each test-tube contains ten beads of the hierarchical color for the order it represents. The first white holder contains test-tubes of green beads, etc.
4. Three division boards, the first one identical to that one used in the memorization of division. The second board has, instead of the green circle 1 in the left hand corner, a blue circle 10. . .and the green strip at the top of the board where the divisor is shown has a blue strip for the digits 1 - 9 instead of the green. The third board has a red circle 100 and a red strip.
4. The box of skittles used with the stamp game, or one identical to it. These are in a rectangular orange box.

First Passage: Long Dividend, One-Digit Divisor

1. The child writes in his notebook the operation: $9764 \div 4 =$

2. He checks to be sure that all the test tubes are full: 10 beads in each.

3. He places the bowls for each hierarchy shown in the dividend above the one board he is using, the first unit board. Then he places behind each of the bowls the test-tube holder. The rest of the bowls and holders are put aside.

HIERARCHICAL DIVISION. . .
First Passage. . .

4. He shows the dividend, with the colored beads from the test-tubes in the holder, in the corresponding bowls. So. . .
   for the 9 thousands, he takes a test-tube from the first grey holder, the only grey holder he has brought forward for this dividend. He knows there are ten beads in the tube; so he takes one out and puts the other nine in the dish, replaces the one bead in the tube and puts it back in the holder. (An excellent clue as to whether the child has understood the concept of 10 here.)

5. On the unit division board he sets up 4 skittles across the top to represent the divisor. He places the board directly under these bowls of beads he has prepared.

6. He begins his division with the largest quantity, distributing the 9 thousands between the 4 skittles.

7. He is able to give 2 thousand beads to each of the 4 skittles.

   He writes the 2 in green in his notebook to signify thousands.

7. I have given each man 2 thousands. I want to know the value of the digit in the quotient; and since I know that these 2 that each man got are thousands, I'll write that digit in green.
   9,764 ÷ 4 = 2

8. Note the remainder of 1 thousand bead. The child replaces that bead in the tube and empties one whole test-tube of hundred beads into the corresponding dish.

8. Not all the thousand beads were distributed. I have one bead remaining. So I exchange that thousand bead for 10 hundred beads.
   I'll need every one in this test-tube, and I'll put them in the hundreds dish.

9. The child replaces all the thousand beads shown on the board in the test-tubes and then moves both the thousands dish and holder aside.

10. He distributes the hundreds between the four skittles on the board. Each man gets 4 ---and he writes that in red: 9,764 ÷ 4 = 24.

11. He notes the remainder of 1, exchanges it for ten tens which he puts in the tens bowl, replaces all the hundred beads in the test-tubes and puts them aside.

12. Repeat for the distribution of tens and units, writing the tens digit in blue and the units digit in green: 9,764 ÷ 4 = 2441

Second Passage: 81 ÷ 8 = : To point out the importance of the value of each digit of the quotient.

1. The child shows the dividend in the two bowls---units and tens. He shows 8 skittles on the unit division board.

2. He distributes the tens and writes a blue 1 in his answer.

3. He sees that he cannot divide the 1.

   Emphasize: We CANNOT skip a hierarchy. We must keep its place with a zero.

3. It is not possible to distribute 1 among 8 skittles.
   I cannot distribute it, so I write a green zero.
   Whenever I consider a hierarchy, even though I cannot distribute it, I must write a zero in the quotient as a place holder. So 81 ÷ 8 = 10 r. 1

Third Passage: Examining Each Part of a Division: Dividend, Divisor, Quotient, Remainder

1. The child writes the division in his notebook: 25, 357 ÷ 6 =

2. He shows the dividend in the dishes.

3. When he begins the distribution, he sees that he cannot divide the two ten thousands between six skittles.

He combines the ten thousands and the thousands beads (exchanging each of the two ten thousands for ten thousands).

3. Here we cannot distribute the first quantity.
BUT WE DON'T WRITE A ZERO AT THE BEGINNING.
Instead we can unite the first two orders.
If we change the two ten thousands into thousands, how many thousands will we have to divide among the 6 skittles?
25
So here we have put both hierarchies together and we have 25 units of thousands.

4. Now he distributes the thousands beads. And he notes the remainder in his written operation below the thousands position.

4. The first digit will be 4 in green.
There is a remainder in the bowl.
We write the remainder like this:

25,357 ÷ 6 = 4
1 3

And then we bring down the digit of the hundreds.
Now the number we have below shows the number of beads we have in the hundreds bowl to distribute.
We have 13 hundreds to divide among the 6 skittles.

5. Exchange that remaining one thousand for ten hundred beads, distribute those hundreds; and show the remainder again in the written operation.
Repeat for the tens and hundreds.

5.

25,357 ÷ 6 = 4236
1 3
15
37
1 R.

We have 15 tens in the bowl to distribute. . .
We have 37 units in the bowl.

6. Note the remainder---write it below the column after the last digit shown.

GROUP DIVISION

We have, to this point, considered division as distribution. Now we move to consider group division, which the child has seen with the stamp game. Group division is important because when division is carried out abstractly, it is considered as group division. And it is necessary to solve some problems with group division.

An example: Distributive: If I have 12 candies to divide among 6 children, how many will each receive?

Group: If I have 20¢ and I want to buy candies, each one being 5¢, how many candies can I buy? (Here I must carry out group division for the answer to find out how many fives are contained in 20)

A PREPARATION: Reviewing group division with the decimal system materials.

Take 25 loose golden beads; and with the child, discover how many groups of five are therein contained.

How many groups of 5 can I form with 25 beads?
Here is one group. . .another group. . .
I have been able to form 5 groups of 5 beads each from 25.

GROUP DIVISION : ONE-DIGIT DIVISOR

NOTE: Here the division process with the long division materials is the same; the difference is the way we verbalize the process.

1. The child writes the division; then forms the dividend with the beads in the four appropriate bowls. And shows the divisor on the unit board with skittles.

1. 9684 ÷ 5 =

2. The child begins by discovering how many groups of five are contained in 9000, distributing the green thousand beads among the five skittles. One complete distribution equals one group.

2. Let's see how many times this group of 5 is contained in 9.
It is contained only one time in 9.
There is only one group of 5 in 9.

3. The child then multiplies the quotient digit times the divisor to find out how many beads he has used. Then he shows that product below the dividend digit.

```
9684 ÷ 5 =
-5
 4
```

3. I want to find out how many of these beads I used.
I multiply the quotient which is the number of groups I have been able to form, times the divisor, which tells me how many are in each group.
And I write that produce below the 9.

4. Then the child subtracts that product from the dividend digit--- THEN HE CHECKS TO SEE WHETHER HE DOES, IN FACT, HAVE THAT MANY BEADS REMAINING IN HIS BOWL.

4. I had 9 units of 1000s, and I used 5.
So I subtract this 5.
Now do I have 4 beads in the bowl?

5. Then the board is cleared of the distributed beads. Then the hundreds digit is brought down in the calculation and the hundreds bowl is brought down to the board. The remaining 4 thousands are then exchanged for hundreds. And the hundreds are distributed in groups of 5.

```
9684 ÷ 5 =
-5
 46
 45
  1
```

5. Now how many hundreds do I have?
How many groups of 5 can be formed with 46?
1 group. . .2 groups. . .3 groups. . .
I have been able to form 9 groups of 5 with 46.
How many hundreds did I use?
How many groups of 5? Then I multiply 9 times 5; and subtract that product from the 46 that I had.
And I should have one bead remaining in the bowl. (He checks)

6. The child clears the board of hundred beads, brings down the tens in the calculation and brings down the tens bowl. Then he exchanges the remaining hundred for one tube of tens. And proceeds to discover how many groups of 5 can be formed of 18 tens.

6. We exchange the remaining hundred for ten 10s.
So I'll have 18 tens in the bowl.
I was able to form three groups of 5 with 18.
How many 10s did I use?
5 X 3 = 15.
Then I should have 3 beads left.
YES--- in the bowl are 3 tens.

7. He now clears the board of tens, brings down the units in the calculation and the units bowl. Makes the exchange for the three tubes of units. . .and discovers the groups of 5 in 37.

7.

NOTE: This is the last passage before abstraction. Before going to two-digit division, the child must have mastered group division.

AGE: 6½

DIRECT AIM: To make the child understand the technique of group division; to bring the child to abstraction.

TWO-DIGIT DIVISION: Long Division (with two or more digits in the divisor)

1. Recall the two-digit division with the decimal system work.

1. Remember the division that we did with the decimal system material?
Your friend wore a blue ribbon.
And the child with the blue ribbon always received 10 times more than the child with the green ribbon. Why?
When a child wore a red ribbon, what did that mean? How much did that child receive?

2. Introduce the second division board, the board of the tens.

2. Each bead that we place on this board represents ten times as much as the bead that we show on our first board. When we distribute one bead on this tens board, we must distribute a bead that has a value of ten times less to the first unit board.

3. The child writes the division. Then he shows the dividend as usual in the bowls with the beads. He shows the blue skittles on the tens board to represent the two tens and the green skittles on the unit board.

3. On the tens board, I will show two blue skittles. On the units board, four green skittles because we now are dividing: 37,464 ÷ 24 =

4. Bring down the tens of thousands(bowl and test-tube holder) above the tens board. And the thousands above the units board.

4. I will give 10,000s to the tens and then how much shall I give to the units?
Ten times less---so I give them 1000s.

5. The child does the distribution, and reads the quotient digit as that number which the UNIT MAN GETS. He writes the quotient digits in colors again here because it is so important that we see the answer as what one man (one unit) receives.

5. I find that I can distribute the 10,000s only once.
I have only one 10,000 left and 2 tens.
Now I have distributed 1,000 to each of the units.
And that is my quotient digit---the amount that one unit gets.
So I WRITE MY QUOTIENT DIGIT IN GREEN.

6. Now the child removes all the beads from both boards, replacing them in the test tubes. He exchanges the remaining 10,000 for 10 thousands and puts the 10,000 bowl and holder aside. THEN he moves the thousands bowls and holder above the tens board and brings the hundreds equipment to the units board.

6. I cannot distribute more 10,000s. So I exchange my one remaining for 1000s. Now if I give units of thousands to the tens, I must give hundreds to the units.

7. He distributes the thousands and hundreds. When there are not enough hundreds, he exchanges one of the thousands for ten hundreds and continues the distribution.

He repeats this process when he again does not have sufficient hundreds.

7. Now I have given 2 thousands to each of the tens, but I have only one row for the hundreds.
And I do not have enough hundreds to continue.
So I must exchange one 1000 for ten more hundreds and I can continue.

8. When this distribution is completed, we read what the unit has received--- 5 hundreds. . .and write the quotient digit in red.

8. Now we have only one unit of thousands left. So we read what the unit has received--- 5 hundreds.
And we write that 5 in our answer in red.

9. He exchanges the remaining thousand for ten hundreds, puts the thousands equipment aside. And distributes hundreds to tens and tens to units.

9. If I give 100s to the tens, I give tens to the units.
When there are no more hundreds, we write the quotient digit in blue because that's what the units have received--- 6 tens.

10. Now he distributes tens to the tens and units to the units, completed the quotient with a green units digit.

$$37{,}464 \div 24 = 1567$$

Distributive Division with a TWO-DIGIT DIVISION, CONSIDERING THE REMAINDER

1. The division:

$$7{,}886 \div 35 = 225 \text{ r. } 11$$

```
 88
 186
 ---
  11
```

1. First we give thousands to the 10s and hundreds to the units.
Our quotient digit is written in red because it is hundreds.
And this time we write in our calculation how many beads remain.
There are 8 hundreds left.
Then we bring down in the calculation the number of tens we show in our dividend. . . and see that now we shall distribute 8 hundreds and 8 tens.

2. We distribute the hundreds to the 10s and the tens to the units.
When we have only one hundred left, we can distribute no more.
The units received 2 tens. I write a blue 2.
And I show that there remains in my bowls one hundred and 8 tens. I write that in the calculation. . .and then bring down the units digit.

3. Now I have 18 tens to distribute to the tens and 6 units to distribute to the units. Some exchanging will be necessary.

GROUP DIVISION With A Two-Digit Divisor

1. Repeat the preparation with the decimal system material. Use 36 loose golden unit beads; and discover how many groups of 12 can be formed with that 36.

2. The division: 8,847 ÷ 24 =
The child shows the dividend as beads in the bowls and the skittles on the two boards, as before. He begins the distribution of thousands to the tens and hundreds to the units, but now he is forming groups.

2. I must discover how many groups of 24 are contained in 88.
I begin by finding out how many groups of 2 are contained in 8.

3. Because he is able to form four groups of 2 with the 8 thousands beads, and only 2 groups of 4 with the 8 hundreds beads; he must exchange some of those beads already shown on the board in order to have adequate hundreds to equalize the distribution, giving as many beads to form the groups of 24 to the units as to the tens. HE REMOVES THEN, ONE GROUP FROM THE TENS, PUTS THEM BACK IN THE BOWL, THEN TAKES THEM ONE AT A TIME TO MAKE 10 MORE HUNDREDS, until he equalizes the groups.

3. Now I want to find out how many groups of 4 there are in 8.
But when I run out of hundreds I have formed only 2 groups.
And here for the tens I have formed 4 groups.
In order to show a whole group of 24, I know I must give the units one bead for each bead I have given the tens.
So I must take one of the tens groups off---that is 2 thousands beads.
And I'll exchange one of those thousands for ten hundreds.
Now I can make another group of 4 for the units.
And I see that I have been able to form 3 groups of 24 with these 88 beads.
My quotient digit will be 3 hundreds.

4. As in the previous group division, he now multiplies the number of groups, the quotient digit, times the divisor and subtracts that product from the dividend. We may note, as he carries over in this multiplication, when he does that we are giving back that ten which we exchanged.

4. I have seen that I can form 3 groups of 24.
So I multiply 24 X 3---and subtract that from the dividend I divided.
Then I should have 16 left--- one thousand bead and six hundreds.
I do.

5. The child exchanges the remaining thousand for ten hundreds. Then he forms groups of 24, giving the tens hundreds and units tens.

5. How many group of 24 can I form with 164?
I begin by discovering how many groups of 2 are contained in 16.
Then I find how many groups of 4 are in 4.
And I see that I must take the last group of hundreds off the board to provide more tens so that I can form as many groups of tens for the units as I have hundreds for the tens.
I know I must have the same number of groups for each to show groups of 24.

He exchanges the groups of hundreds, taking one group off the board, then exchanging the beads one at a time, until there are the same number of groups for tens and units.

6. Repeat for the final distribution, forming groups of 24 with the tens to the tens and the units to the units.

8847 ÷ 24 = 368 r. 15

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8847
-72
 164
-144
  207
 -192
   15
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DIVISION WITH A 3-DIGIT DIVISOR: LONG DIVISION

The child must know how to carry out two-digit divisor division abstractly before going onto longer division with the three-digit divisor. When the child reaches abstraction, he abandons the material; at this point we must offer new material to recreate his interest. But, when he understands the mechanics of 2-digit divisor division well and abstractly, he meets the 3-digit divisor easily and will not be interested in working with the material long. In the work with the 3-digit divisor we skip distributive division because the child is working abstractly with division. So we begin with Group Division.

Long Division: Group Division: 56,438 ÷ 234 =

1. Introduce the third division board, that board of the hundreds. The child forms the dividend in the corresponding bowls. Then he shows the divisor, this time on all three boards: 2 red skittles on the hundreds board, 3 blue skittles on the tens board, 4 green skittles on the units board.

2. He begins by distributing the ten-thousands to the hundreds, the thousands to the tens, the hundreds to the units. IN GROUPS.

2. I want to know how many times 234 is contained (goes) in 564.
The 2 goes into the 5 2 times.
Now. . .does the 3 also go into the 6 two times? YES.
Will the 4 go into the 4 two times? NO.

3. The child here must change a ten thousand bead for ten thousand beads because there are none. Then he can exchange one thousand for 10 tens to complete two groups for the units.

3. I have no thousands to borrow.
So I change a ten thousand for 10 thousands, and then I can change 1 thousand for 10 hundreds.
Now with the hundreds I can form 2 groups for the units.

4. When the groups have been equalized and shown, the child multiplies the number of groups times the divisor and then subtracts the product from the dividend figure he was dividing.
HE CHECKS TO SEE THAT HIS DIFFERENCE IS, IN FACT, THE BEADS THAT REMAIN.

4. Now I can say that the whole group of 234 goes into 564 two times. I write that 2 as my quotient digit.
If my operation is correct, I should have 9 thousands beads and 6 hundreds beads remaining.

5. Then the child eliminates the ten thousand equipment, shows the thousands above the hundreds board, the hundreds above the tens board, the tens above the units.
He distributes in groups: there are 4 groups of thousands on the first board; he changes 1 thousand for 10 hundreds to make 4 groups on the second board; then, not being able to form 4 groups of tens for the units, he changes 1 hundred for 10 tens. That makes four groups for all. He has shown that 234 goes into 963 four times. He writes the quotient digit; multiplies, then subtracts from the dividend. CHECKS to see if 2 hundred beads and 7 tens remain. He replaces the beads in the tubes, eliminates the thousands materials, moves all the hierarchies up one board. And proceeds as before.

56,438 ÷ 234 = 241 r. 44
−468
= 963
− 936
= 278
− 234
44

6. In completing the operation, 4 tens and 4 units remain. We cannot distribute these because there would not be another group for the hundreds. So 44 is the remainder.

HIERARCHICAL DIVISION. . .
3-Digit Divisor. . .

SPECIAL CASES: Case #1: Zero in the Place of the Tens in the Divisor
Zero in the Place of the Units in the Divisor
Zero in the Place of the Tens and the Units in the Divisor

The child has met these special cases before. He has seen a divisor of 100 in centurion division with the decimal system materials. He has met these special cases in the stamp game. Now he recalls the previous experiences in a new way.

Presentation: Case #1: 51,252 ÷ 207 =

1. The child forms the dividend as usual and the divisor on two of the three boards displayed. We must point out that, even though there is a zero in the place of the tens, we cannot skip the hierarchy. The board remains to show its place.

2. The child first discovers how many times 207 goes into 512. The hundreds receive the ten thousands and the units receive the hundreds. But we do not distribute anything to the tens. However, that dividend digit is shown in the beads in the bowl above the tens board. It marks the hierarchy. It is used if more hundreds are needed by the units. OR it simply stands silently until the thousands are moving up to the next board for distribution to the hundreds.

3. Then when the thousands are distributed to the hundreds, the tens are distributed to the units. Now the hundreds keep the place above the tens board.

4. When we divide 207 into 1,572 (the remainder shown after the second subtraction), we discover that we must change that thousand so that we have 15 hundreds with which to form groups of 2 for the hundreds. The 7 tens rest until needed for the exchange and we form groups of 7 with the 2 units. (necessitating exchanges)

51,252 ÷ 207 = 247 r.123
−414
= 985
−828
1572
−1449
= 123

Presentation: Case #2: 19,293 ÷ 370 =
In our first experience with this case, we return to the quotient written in the hierarchical colors.

1. We must begin by finding how many groups of 370 are contained in 192, an obviously impossible division. So we must take into consideration the next digit. When we have exchanged the ten thousand bead for 10 thousands, we remove the ten thousand equipment and give thousands to the hundreds, hundreds to the tens and tens are placed above the units---BUT NONE DISTRIBUTED.

2. In this formation of the groups, we cannot form as many groups of hundreds as we have formed groups of thousands. So we must remove a whole group from the hundreds board in order to add to our hundreds beads. Then we exchange the thousand beads one at a time until we have enough hundreds to form as many groups as we have formed for the thousands.

3. When the first groups of 270 have been formed from 1929 (5 groups), we consider the quotient digit and its color. We have formed 5 groups. But we write 5 groups of what the units receive. That would have been tens, so we write the digit in blue. The answer is what the units receive. We are actually dividing the number of groups on the tens board by 10 because we know that the units will have received ten times less than the tens.

4. For the same reason, in the second part of the operation, the quotient digit will be green.

19,293 ÷ 370 = 52 r.53
−1850
= 793
740
53