FRACTIONS. . .

Presentation: Numerator and Denominator. . .

1. . .

It is simply a division.
A division is simply a fraction.

2. Use the blackboard to demonstrate the progression of signs used.

2. The Latins wrote divisions in this way:
To write 8 divided by 4 they wrote

8 Fractus 4

Then at a certain point they got tired of writing the whole word and wrote:

8 F 4

Then they got tired of writing the whole letter and wrote

8 ⊏ 4

Now we just write

8 ÷ 4

Man is always looking for the fastest way to do things.
AND - that is why fractions are written this way.

$4/8 \qquad \frac{4}{8}$

It doesn't matter which way the line goes. It is a division.
When I break the whole into fractions I divide it.

3. Introduce the words numerator and denominator.

3. Now I will tell you how to write all fractions.
Here is the fourths family:

$\frac{1}{4} \; \frac{2}{4} \; \frac{3}{4} \; \frac{4}{4}$

In working with fractions, this last name is called the denominator.
And this number of members or parts that we take is called the numerator.

4. Use the three-period lesson.

4. Show me the dividing line.
Show me the numerator.
What is this?

FRACTIONS. . .

Equivalence

Material

1. Only the circle fractional insets. (a second hard plastic set, showing the whole circle and then the circle divided into fractions from halves to tenths.
2. THE GEOMETRY CHARTS: the circle charts which correspond to the points of presentation made. These charts should be hung in the classroom after the presentation.

Presentation

1. With the set of circle insets displayed, remove the whole from the frame and replace it with the two halves.	1. These two halves occupy exactly the same space as the whole.
2. Do the same demonstration with all the fractional parts to show that each group occupies the same space---although all the fractional parts are different. . .that they are not equal, but equivalent.	2. Now let's try the thirds. These three thirds also occupy the same space as the whole. Is the third the same size as the half? How many halves did we need to occupy the same space as a whole? How many thirds? These four fourths also occupy the same space as the whole.
3. Show the other whole circle inset from the first metal set. Fit it in the first frame. Both are the same---equal. Emphasize the important concept of the different between equal and equivalent.	3. When two figures are equal: when one is placed on top of another and they are shown to be exactly alike---those figres are EQUAL. The two halves are not the same as the whole. They are NOT equal. But they show equivalence. . . When the figures are not equal, but they occupy the same space, they are called EQUIVALENTS. Equivalent comes from the Latin word "aeguus," meaning "the same" and "valere" meaning "value."
4. Introduce the sign for equivalence as contrasted to the sign of equality.	4. The sign of equivalence is three lines: $\equiv$. What is the sign for equal? It is important to remember that these signs mean different things.

NOTE: The difference between similar, equivalent and equal is an important one as well as the signs for each of the three. We shall emphasize the point in further presentations. Here only the difference between equal and equivalent is established. $=$ $\equiv$ $\sim$

Equivalence. . .
Presentation. . .

NOTE: The sign of equivalence is used only in geometry and not with the arithmetical operations. So. . .

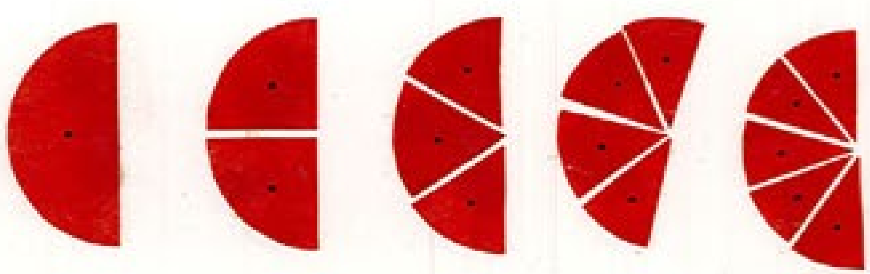
for $\frac{2}{7} + \frac{5}{7} = \frac{7}{7}$ we use the equal sign.

5. Refer to the previous demonstration of equivalence. . .

5. Therefore, we can say that two halves are equivalent to one whole;
Three thirds are equivalent to one whole;
Four fourths;

6. Remove one of the two halves. Discover the equivalences with the child.

6. Now let's look for other equivalences.
Let's see what other fraction we can put in this space.
We see that it is not possible with the thirds.
No combination of thirds will occupy the same space as one half.
It is possible to fit twof ourths into this space.
This means that ½ is equivalent to 2/4s---or vice versa.
With fifths it is not possible.
With sixths it is.
This means that ½ is equivalent to 3/6s.
½ is also equivalent to 4/8s and 5/10s.



$\equiv \frac{1}{2}$

7. The child can draw the equivalent fractions or cut them out of red paper for pasting in their notebooks. He writes:

1/2 ≡ 2/4 ≡ 3/6 ≡ 4/8 ≡ 5/10 OR 3/6 ≡ 1/2

Equivalence. . .
Presentation. . .

8. Present Circle Charts #1, #2, and #3.

9. The child can look for more equivalences on his own. He may remove the 1/3 and look for the fractions equivalent. Or 2/7,

OR

10. Using the commands prepared by the teacher, the child may: Look for the equivalent fractions for 2/3, for 1/4.

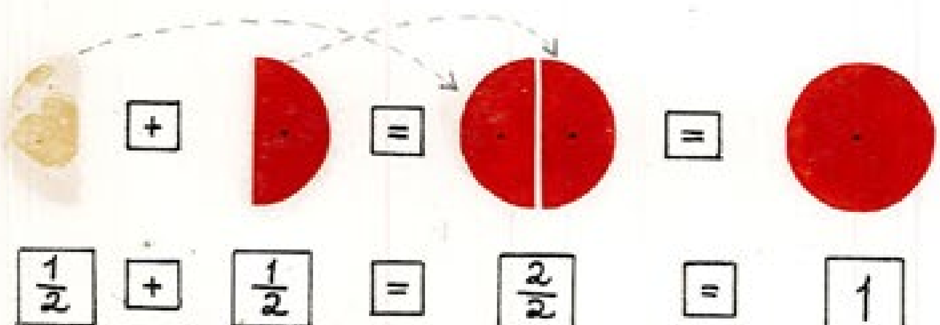
AGE: 7-8

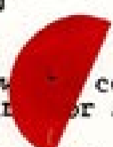
THE OPERATIONS

Material (the same)

Presentation: Addition

1. Introduce addition with fractions----note that with the arithmetical operations, we use the equals sign.	1. Now we are going to make arithmetical operations with fractions. So we will not use the equivalence sign for geometry with the three lines. We will use our equals sign. . .=
2. Begin the addition with 2 halves. Show the addition both with the materials and with written slips.	2. We want to add two halves. We'll use the addition sign. We are putting the two fractions together. That is equal to 2/2. 2/2 is equal to 1.





3. Show the corresponding circle chart for addition.

4. Show another example of addition, using both paper slips and materials. 1/5 + 3/5 = 4/5

5. The child can color or cut out the fractions for an addition, writing the whole operation in his notebook.

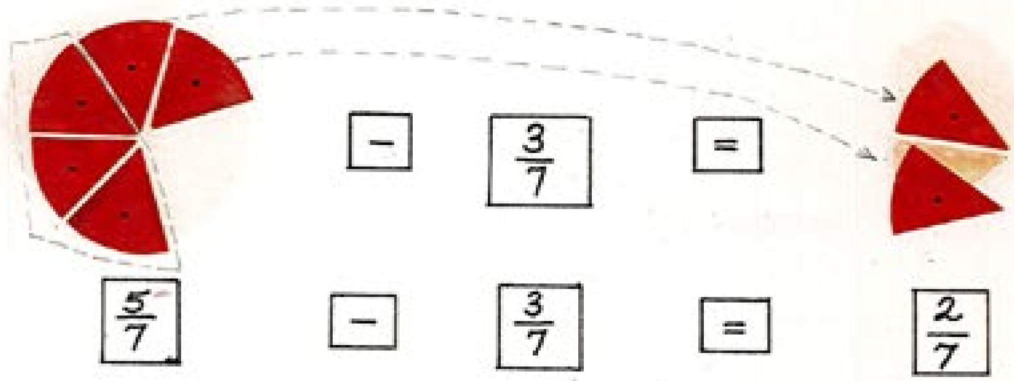
FRACTIONS. . .The Operations. . .

Presentation: Subtraction

1. Write on slips and display on the mat: 5/7 - 3/7 = Ask the child to set up the material---to see whether he knows to only show the minuend.

1. I have 5/7.
If I take away 3/7, I have 2/7 left.

We can show those 2/7 after the equals sign.
That is our answer: we have 2/7 left.



2. Read the subtraction carefully with the child---and note the process. Then show an addition and read, noting the change again only in the numerator.

2. Read: 1/5 + 3/5 = 4/5
5/7 - 3/7 = 2/7

What do you do when you subtract fractions with the same denominator?
What is the quantity that changes in addition?
Which one remains the same?
The one above the line changes: the numerator.
The denominator doesn't change.
So---what happens in both addition and subtraction is that the numerator changes and the denominator stays the same.

3. Formulate a rule. . .not for the child to memorize, but to understand.

3. Let's form a rule: When we add or subtract fractions with the same denominator, we add or subtract the numerator and the denominator stays the same.

4. Display Chart #8 which states this rule---and Chart #12, a subtraction with the same denominator.

FRACTIONS: The Operations

Presentation: Multiplication of a Fraction times a Whole Number.

1. Write on slips: a multiplication and ask the child to place the corresponding material.

   He shows the 2/5 four times. He may use the insets or, if enough sets are not available, we may prepare others cut.

1. What does it mean to multiply? It means "to take." If I want to multiply, 2/5 X 4, what shall I do?

   How many times do I take the 2/5? This number tells me I have to take this fraction 4 times. Then we will have to show 2/5 four times.

2. If the product is more than one whole, the child first shows all the fractional parts, with a corresponding slip; and then he substitutes the whole plus the fractional part as a final product.

   And write a corresponding slip.

2. Then what is our product? It is 8/5. Oh---but look what has happened. We have more than 1. We have 1 whole and 3/5. How can we show that with the insets? We will use 1 whole and three fifths. We write that: 1 3/5.

3. Present chart #14.

Presentation: Another example of multiplication: Division of denominator

1. The problem is presented on the mat and the child forms the multiplication with the material.

1. $\frac{1}{4}$ X 2 = 2/4 = $\frac{1}{2}$

2. Note that the numerator changes and the denominator stays the same.

2. What do you do when you multiply a fraction times a whole? We multiply the numerator and the denominator stays the same. Here we multiplied 2 X 1 (fourth) and our result was 2/4 or 1/2.

3. When the children have worked with the operations a long time, we can make this explanation for him, helping him towards abstraction.

4. We discover a new rule of multiplication.

   To multiply a fraction by a whole number, one can multiply the numerator by the whole number, OR, if possible (the denominator divisible), we may divide the denominator by the whole number.

   SHOW THE CORRESPONDING CIRCLE CHART WITH THE RULE.

4. How we want to discover a faster way to arrive at this 1/2. When we did our operation:

$$\frac{1}{4} \times 2 = \frac{1 \times 2}{4} = \frac{2}{4} = \frac{1}{2}$$

Now we can get the same result by working with the denominator:

$$\frac{1}{4} \times 2 = \frac{1}{4 \div 2} = \frac{1}{2}$$

The numerator stays the same and the denominator is divided by the whole number.

Addition and Subtraction: Moving Towards Abstraction

Presentation: After the child has worked with the fractional materials with the operations of addition and subtraction for a long time.

NOTE: The child may, at a certain point in his work with the materials, discover on his own what happens to the fractions with which he is working in addition and subtraction. If not, then we must help him towards abstraction with the following explanation and the formulation of the RULE.

1. Show the child that:

   Write the calculation, drawing the line through the 3 to show that it has not been involved in the addition--- it stays the same.

1. When we add the fractions one-third and two-thirds, what is our answer: three-thirds or one whole.
   Let's see what has happened:

   $$\frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1$$

   $$\frac{1}{3} + \frac{2}{3} = \frac{1+2}{\not{3}} = \frac{3}{3} = 1$$

   We have added the two numerators, but the denominator is unchanged. It remains 3. . .throughout the operation.

2. Show the same written calculation with a subtraction.

3. Help the child formulate the rule.

3. Let's see if we can discover a rule: If I must add or subtract fractions which have the same denominators, I leave the denominator the same and perform the operation on the numerator.

4. Present circle chart #8: the rule.

Multiplication: Moving Towards Abstraction

1. Show that in multiplication, when a fraction is multiplied by a whole number, the denominator remains unchanged and the operation is performed on the numerator.

1. Now let's see what happens in the multiplication of a fraction by a whole number:

   $$\frac{2}{7} \times 3 = \frac{2 \times 3}{\not{7}} = \frac{6}{7}$$

   It is the 2 that is multiplied by the 3. The 7 denominator of the product is the same as the denominator throughout.

FRACTIONS. . .

Equivalence: Moving Towards Abstraction

Presentation: Made when the child has worked with the equivalences of the fractional materials for about a year. To bring him to abstraction so that he may continue without the materials.

1. Show the $\frac{1}{2}$ and the 2/4 in the whole inset.

1. What equivalence is shown here? What has happened here? We see that:

$$\frac{1}{2} = \underline{\qquad\qquad} = \frac{2}{4}$$

and

$$\frac{2}{4} = \underline{\qquad\qquad} = \frac{1}{2}$$

2. Show the child that the result in both cases can be arrived at by multiplication of the numerator and the denominator by the same number.

2. If I wish to transform this without the material, what operation can I do to arrive at this result? In the first case, the 1 has become a 2; and the 2 has become a 4. So:

$$\frac{1}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4}$$

This half, which was 1 piece, has become two pieces. But the two pieces are not called halves: they are fourths.
What have I done?
I have multiplied the 1 by 2 and the 2 by 2.

It is important here---and later ---that the child knows that multiplication means "to take," that number a certain number of times.

3. Show the division of both numerato and denominator by the same number to obtain the second equivalence.

In the second case, 2 pieces have become 1 piece. What have we done?

$$\frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2}$$

We have divided the 2 by 2 and the 4 by 2.

4. Note that we do not change the value of the fraction when multiplying or dividing both terms by the same number.

4. We have said that equivalents occupy the same space. They have the same value. If I eat ½ a pie or 2/4 of a pie, I eat the same amount. Here I've multiplied each term by 2, here I've divided each term by 2---and the value is unchanged.

5. Formulate the rule.

5. Multiplying and dividing a fraction (both the numerator and the denominator) by the same number does not change the value of the fraction.

6. Now the child is ready to do all the equivalence exercises which he did with the material WITHOUT the material.

He begins with the equivalence:

$$\frac{1}{2} = \underline{\qquad\qquad} = \frac{3}{6} \qquad\qquad \frac{3}{6} = \underline{\qquad\qquad} = \frac{1}{2}$$

And then he discovers what has happened---and through this process he proves the rule.

$$\frac{1}{2} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6} \qquad\qquad \frac{3}{6} = \frac{3 \div 3}{6 \div 3} = \frac{1}{2}$$

FRACTIONS. . .

Division of a Fraction By a Whole Number

Material

1. The same fraction insets. (circles)
2. The division skittles.

Presentation

1. Write the division on a slip. NOTE: all division problems must be prepared to avoid difficulties at this point.	1. $9/10 \div 3 =$
2. The child places the fraction materials out and then, to the right of the material, shows the skittles as the divisor.	2. Now I must show nine tenths. And that quantity must be divided equally among 3 skittles.
3. The child distributes the fractions . . .note the result.	3. What is our result? It is what one unit receives. That is equal to 3/10.
4. Show chart #18: this problem.	

5. WHEN THE CHILD HAS WORKED WITH THE MATERIAL A LONG TIME IN THIS WAY:

we show him what has happened:

$$\frac{9}{10} \div 3 = \text{———} \; \frac{3}{10} \qquad \frac{9}{10} \div 3 = \frac{9 \div 3}{10} = \frac{3}{10}$$

and formulate the rule: To divide a fraction by a whole number, we leave the denominator unchanged and divide the numerator by the whole number.

AGE: The fractional work to this point: addition and subtraction with the same denominators, multiplication and division with whole numbers, constitutes the whole of that work presented to the child through the age of 8----during the years 7 and 8 he meets fractions in this way and also gains a good understanding of equivalence.

Presentation #1: A Preparation: Nomenclature (Using the circle insets)

1. Explain the concepts of real, apparent, and improper fractions:
   There are two different kinds of fractions. 2/4 is a REAL FRACTION because it names a quantity which is smaller than the unit. (SHOW two fourths taken from the circle inset of fourths)
   But if we say 4/4, we have named one whole which is divided into four parts. When we take the parts together we have the whole. 4/4 is called an APPARENT FRACTION. (SHOW the four fourths together in the circle inset.)
   Now let's take 6/4. (USE the four fourths from the inset + two red cardboard fourths.) 6/4 is not smaller than the whole; AND it is not THE whole. It is larger than the whole. We call it an IMPROPER FRACTION. (SHOW the six fourths.)

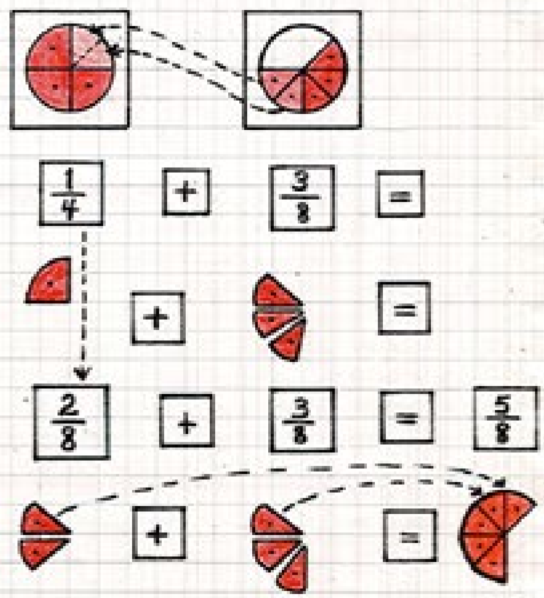
2. Ask the child to form examples of real, apparent and improper fractions with the parts from the circle insets. He identifies them.

3. Three-period lesson.

WHEN THE CHILD HAS UNDERSTOOD THIS WELL:

Presentation #2: Addition with Fractions of Different Denominators

1. Introduce the concept of transforming two denominators into the same family:
   If we want to add peas and beans, in order to add them we can't put them all together to count them; but we must first group the peas and then group the beans. They cannot be added all together because they are not in the same family.
   The same is true with fractions: we cannot add 1/4 and 3/8 because they do not belong to the same family. The 1/4 belongs to the family of fourths; the 3/8 belong to the family of eighths. We must try to transform the eighths into fourths or the fourths into eighths in order to made an addition (or subtraction) with members of the same family.



2. Show the proposed addition with the materials and the symbols.

3. Leaving the empty frames above the displayed materials, try first to fit the fourths into the space vacated by the 3/8---THEY WON'T GO. Then try eights in the empty space in the frame of the fourths. We see that 2/8 is equal to 1/4. Therefore, we can substitute those 2/8 for the 1/4 in order to make our addition.

4. SUBSITUTE the 2/8 for the 1/4: TURN THE SLIP THAT READS 1/4 OVER and write 2/8.

5. The child makes the addition. He writes the answer 5/8 on a slip and shows it with the materials. Then he copies the operation in his notebook.

6. Problem #2: 2/4 + 4/8 =
   In this work the child discovers that he can transform the eights into fourths OR he can transform the fourths into eights. He begins by showing the two fourths which remain in the frame after he has set out the materials of the problem in the empty space left in the frame of the eighths. TWO FOURTHS FIT. . .THEREFORE HE CAN SUBSTITUTE 2/4 for the 4/8, write the new fraction on the back of the slip that said 4/8, and make the addition. The result is 4/4 or 1, so he substitutes the WHOLE CIRCLE for the 4/4.
   THEN he returns the materials of the problem to their original positions and tries the reverse process, discovering that he can place 4/8 in the empty space left in the frame of fourths. He makes the necessary substitution with the material, changes the symbol and makes the addition. Again the result if 8/8 or one whole.
   In both cases, the result is the same---1. AND this is an APPARENT FRACTION.

FRACTIONS: Addition and Subtraction with Different Denominators. . .
Addition. . .

7. Problem #3: 1/2 + 2/3 =

We begin the solution for the search of a common family by trying to fit the 1/2 left in the frame into the space vacated by the 2/3 in the thirds frame; then trying the third in the halves frame. NEITHER ONE OCCUPIES THE SPACE EXACTLY. So we must find a new fraction which will fit in place of both the fractions which have been taken out of the frames for our addition. We must find a MEDIATOR which will give us a common family.

The child first tries the fourths. He can fit two fourths exactly into the space in the frame of the halves; but the fourths will not fit in the place of the 2/3. He tries fifths. Then he tries sixths---THEY WORK. He discovers that 3/6 will occupy the space of 1/2 in the frame; and 4/6 will occupy the space vacated by the 2/3.

So he SUBSTITUTES 3/6 (inset sixths) for the 1/2 he has shown in his operation display of materials and symbols: and he writes 3/6 on the back of the slip reading 1/2. Then he substitutes 4/6 for the 2/3 in the same way.

And he makes the addition. His answer is 7/6. IT IS MORE THAN ONE WHOLE. So he substitutes the 7/6 shown with the whole circle inset and one sixth. This then is an improper fraction.

NOTE: Whenever more fractional parts are needed than are available in the actual insets, we use equal cardboard fractions.

8. Problem #4: 1/4 + 1/4 + 3/9 =

The child begins by showing the addition with materials from the insets and corresponding symbols in the usual way.

The first step here is to combine the fractions with the same denominator. Thus we reduce the three fractions to two. The child changes the slips reading 1/4 and 1/4 to one reading 2/4; and combines the two fractional pieces.

Now the addition is 2/4 and 3/9.

He proceeds to find a fraction which will exactly occupy the space of BOTH 2/4 and 3/9. He tries first the fourths in the ninths frame and the ninths in the fourths frame. Neither work. Then he tries fifths. Then sixths---THEY WORK.

He substitutes 3/6 for 2/4; and 2/6 for 3/9, writing the new fraction symbols on the back of both slips. He makes the addition---5/6. It is a real fraction.

NOTE: It is important each time to identify the kind of fraction which is the answer. It is also important that he try the various fractions in his search for the common family in numerical succession---A PREPARATION FOR FINDING THE LOWEST COMMON DENOMINATOR.

9. Problem #5: 1/3 + 3/9 + 2/6 =

This is a particularly interesting exercise. The child finds that he can make the necessary transformation to a common family by using, successively, each of the three denominators. He first discovers that he can convert all the fractions into thirds; then he replaces the original material and tries the ninths in both the frame of the thirds and the sixths. He finds that each can also be transformed into ninths. And finally, he tries the sixths and discovers that all can be shown as members of the sixths family. Each time he completes the addition, he copies it in his notebook; thus he sees that in each of the three operations his result is the same: the whole inset---1.

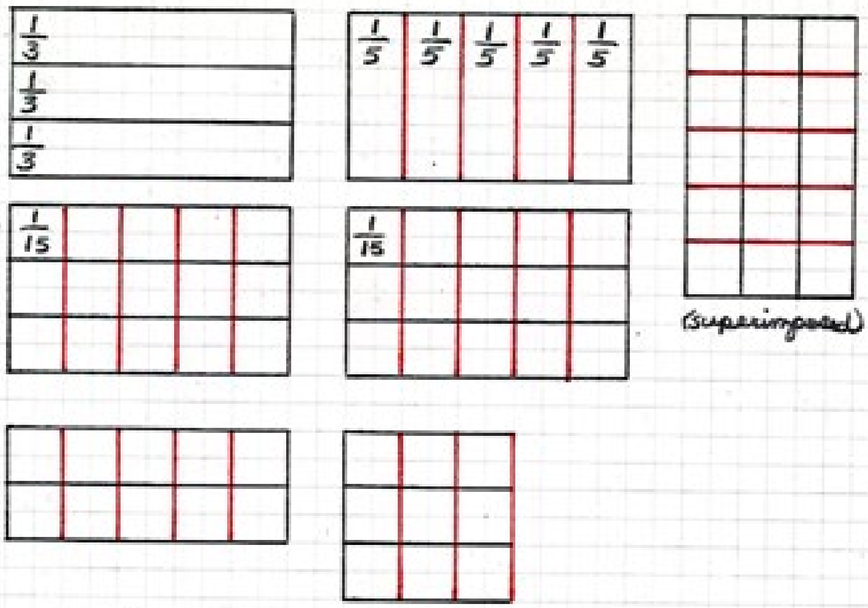
10. Problem #6: 1/2 + 2/5 =

Here the increased difficulty is the long search for the common denominator. The child discovers that the halves will not occupy the space left by 2/5; and that the fifths will not occupy exactly the space vacated by the 1/2. So he tries thirds, fourths, sixths, sevenths, eighths, ninths, and finally tenths. THE TENTHS WORK. He substitutes 5/10 for the 1/2 and 4/10 for the 2/5. And makes the addition: 9/10.

NOTE: Use the circle charts for addition #9 and #10.

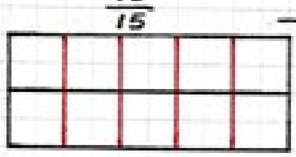
Fractions...
The Concept of Least Common Denominator...

Subtraction $\frac{2}{3} - \frac{3}{5}$



$\frac{2}{3} \quad - \quad \frac{3}{5} =$

$\frac{10}{15}$ –



$\frac{9}{15} = \square \frac{1}{15}$

Process: We take two wholes of the same size. One is divided into thirds and the other into fifths; and then the two are superimposed. The figure is thus shown divided into fifteen equal parts. 15 is therefore the new common denominator. From this whole divided into fifteenths, we take 2/3, that is 10/15; and we show 3/5, that is 9/15. From the minuend which is the 10/15 we subtract 9/16 and we obtain a difference of 1/15.

Abstract operation:

$\frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$; $\frac{3}{5} = \frac{3 \times 3}{5 \times 3} = \frac{9}{15}$;

$\frac{10-9}{15} = \frac{1}{15}$

Chart #2:

When the child does this work, he can use the squares of graph paper as his measure, or he may establish his own measure with a ruler.

Very quickly he sees that the dimensions of his figure are given by the product of the denominators: 3 × 5.

And that only when this common measure is established between the two is he able to take equivalent areas for both fractions.

In his calculation then, he writes as usual the original fraction and the equivalent fraction. And then he discovers the passage.

$\frac{2}{3} = \frac{?}{?} = \frac{10}{15}$

Finding the Least Common Denominator

In this final passage the child no longer uses the material. He first proves that his work with the squares gave the Least Common Denominator by finding the <u>LCM</u> of the denominators 3 & 4.

$\frac{2}{3} + \frac{3}{4} =$ 3 is a prime number: it remains 3

①
$$\begin{array}{r|l} 4 & 2 \\ 2 & 2 \\ 1 & \end{array}$$
$4 = 2^2$ So LCM $= 3 \times 2^2 =$ <u>12</u> = LCD

We could not have made a smaller drawing!

Fractions...
Least Common Denominator...

With 12 established as the Least Common Denominator, we make the following calculation to prove our operation:

$12 \div 3 = 4$ so we multiply both denominator and numerator times 4 $\quad \frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$

$12 \div 4 = 3$ we multiply both times 3 $\quad \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12} = 1\frac{5}{12}$$

② $\frac{2}{3} - \frac{3}{5} =$

3 and 5 are both prime numbers so LCM = 3x5 = 15 = LCD

$15 \div 3 = 5 \quad \frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15} \qquad 15 \div 5 = 3 \quad \frac{3}{5} = \frac{3 \times 3}{5 \times 3} = \frac{9}{15}$

$$\frac{2}{3} - \frac{3}{5} = \frac{10}{15} - \frac{9}{15} = \frac{1}{15}$$

③ Three terms: $\frac{3}{4} + \frac{5}{6} + \frac{7}{8} =$

4	2	6	3	8	2		
2	2	2	2	4	2		
1		1		2	2		
				1			

LCM = $2^3 \times 3$ = 24 = LCD

$24 \div 4 = 6 \rightarrow \frac{3}{4} = \frac{3 \times 6}{4 \times 6} = \frac{18}{24}$

$24 \div 6 = 4 \quad \frac{5}{6} = \frac{5 \times 4}{6 \times 4} = \frac{20}{24} \qquad \frac{3}{4} + \frac{5}{6} + \frac{7}{8} = \frac{18}{24} + \frac{20}{24} + \frac{21}{24} = \frac{59}{24}$

$24 \div 8 = 3 \quad \frac{7}{8} = \frac{7 \times 3}{8 \times 3} = \frac{21}{24}$

MULTIPLICATION AND DIVISION OF A WHOLE NUMBER BY A FRACTION

In multiplication of a fraction by a whole number such as 2/6 X 2, we mean to take 2/6 TWO TIMES. To divide 4/6 by 2 (4/6 ÷ 2) means to take that 4/6 and divide them between TWO (skittles, persons, etc.) Both operations become more difficult when we do the opposite process: when we have a multiplier or a divisor that is a fraction. So we first consider the multiplication and division of the fraction by the whole number at the first level of the fraction work. Here we meet the sedond, more complex, cases.

Presentation #1: Multiplication of a Whole Number by a Fraction

Materials
1. The circle insets.
2. Additional cardboard inset pieces for all the fractions of the circle.
3. The green unit stamps.
4. Slips of paper.

1. As a preparation for the new work, present the multiplication 2 X 2 = . The child uses the whole red unit circles to show the produce on the mat, showing the corresponding notation on slips of paper and using the operation symbols. He copies the operation in his notebook.

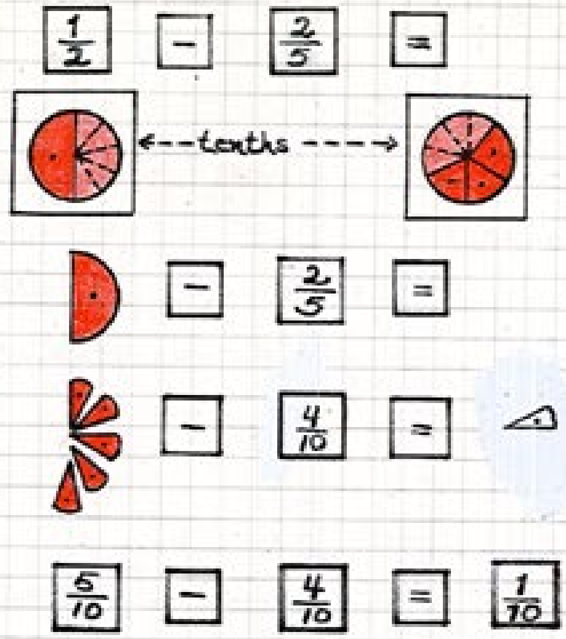
   2 X 2 = means to take the 2 two times.

2. Proceed to 2 X 1 = . The child again shows the multiplication. . .with the cardboard circle insets and symbols. He copies the operation.

   2 X 1 = means to take the 2 one time.

FRACTIONS: **Addition** and Subtraction with Different Denominators. . .

Presentation #3: **Subtraction with Fractions of Different Denominators. . .**



1. Present the problem of subtraction on the mat with the symbols for the child and ask him to form the problem with the materials of the circle insets. HE WILL FORM ONLY THE MINUEND.

2. Introduce the frame of the subtrahend and put the fractional parts aside which represent the subtrahend explaining that **because we have fractions with denominators of different families, we must find a fraction which will occupy the space left in the frames by both terms of the operation. Thus we need the empty space that the subtrahend would occupy, even though we do not need the materials themselves.**

3. Now the child proceeds, as in addition, to find a common denominator for both minuend and subtrahend fraction. Halves will not fit in the fifths frame; fifths will not fit in the halves frame. He tries the successive fractions and discovers that tenths work.

4. He substitutes the materials and changes the written symbols. Then makes the subtraction.

PROCEED TO OTHER EXAMPLES. . .

5. AT THIS POINT THE CHILD SHOULD HAVE BEGUN TO UNDERSTAND THE TRANSFORMATION OF THE FRACTIONS. Recalling the **equivalence work,** we show him how to write the transformation, beginning with the original fraction and the end fraction he has made with the materials. Then deducing the mediator's process. **What did we do to obtain 5/10? 4/10?**

6. Repeat several of the addition problems, WRITING NOW THE TRANSFORMATION. And use other subtraction problems in the same way. INCLUDE TRANS* FORMATIONS WHICH INCLUDE BOTH MULTIPLICATION AND DIVISION BY THE SAME NUMBER.

7. The child does many of these operations in order to help him **deduce the rules.** When the children deduce the rules themselves, they don't forget them because they have understood.

TO ABSTRACTION. . .IN + AND −

$$\frac{1}{2} = \underline{\qquad\qquad} = \frac{5}{10}$$

$$\frac{1}{2} = \frac{1 \times 5}{2 \times 5} = \frac{5}{10}$$

$$\frac{1}{4} + \frac{3}{8} = \frac{2}{8} + \frac{3}{8}$$

$$\frac{1}{4} = \frac{1 \times 2}{4 \times 2} = \frac{2}{8}$$

$$\frac{2}{4} + \frac{4}{8} = \frac{4}{8} + \frac{4}{8} \text{ OR } = \frac{2}{4} + \frac{2}{4}$$

$$\frac{2}{4} = \frac{2 \times 2}{4 \times 2} = \frac{4}{8}$$

$$\frac{4}{8} = \frac{4 \div 2}{8 \div 2} = \frac{2}{4}$$

1. **In order to add or subtract fractions with different denominators, it is necessary to transform fractions to the same denominator before executing the operation.**
2. **In order to change the terms of fractions, it is necessary to multiply or divide the terms by the same number.**
3. **Multiplying or dividing the terms of a fraction by the same number does not change the value of the fraction.**

NOTE: The Rules are found on circle chart #11. See also subtaction charts #12 and #13.

Fractions...

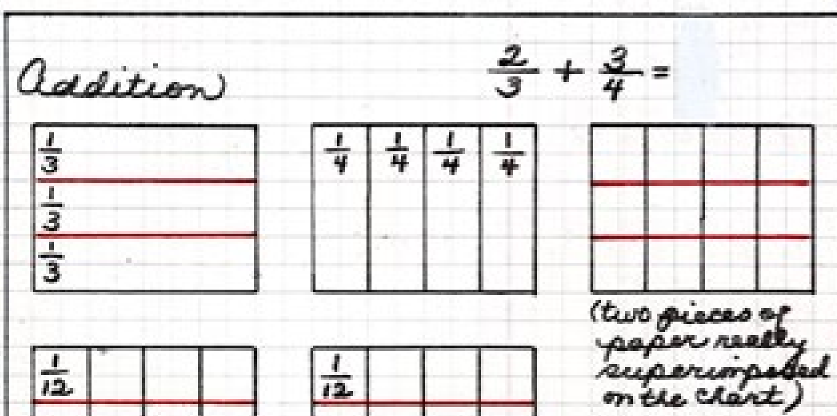
Presentation #4: Passage to Abstraction: Approaching the concept of Lowest Common Denominator

At a certain point we must say to the child that in order to perform the operations of addition and subtraction with fractions, we must transform them to fractions with the same denominator.

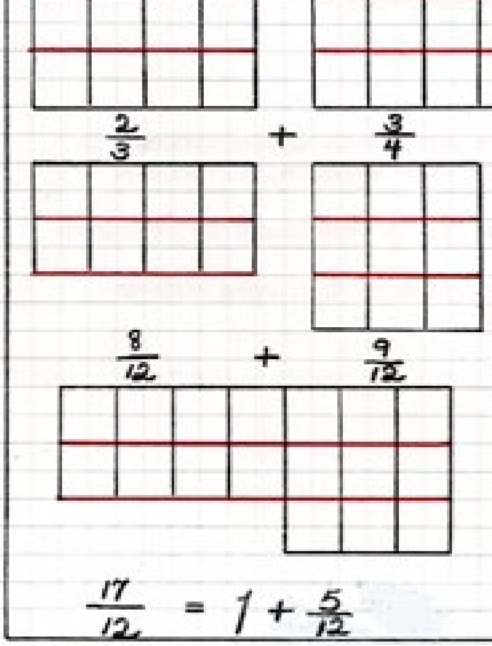
But ... we do not transform them to fractions of any denominator that is a common multiple... we must find the Least Common Denominator. This is the reason why, when searching for equivalent fractions to fill the empty space of each frame (vacated by each term of the operation), we began with the smallest possibilities: so we tried the thirds, then the fourths, then the fifths, etc.

We present the abstract discovery of the Least Common Denominator first with 2 charts: the child may duplicate this work on graph paper.

Chart #1



We consider first only the denominators by which we subdivide our two equal wholes. If, then, we were to use only one subdivided whole as shown in the super-imposed figure, we would first take 8/12... leaving only 4 and we need 9/12 for the second addend. This is an indication that our sum will be greater than 1. ...and we would add the 5 squares.



Process: We take two wholes of the same size. One is divided into thirds and the other into fourths; thereafter the two are superimposed as opposites. The figure is thus shown divided into 12 equal parts. 12 is therefore the new common denominator. From this whole divided into twelfths, we take 2/3, that is, 8/12; and 3/4, that is, 9/12. Adding the two, we obtain 17/12 = 1 5/12.

Abstract operation:

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}\ ;\ \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}\ ;$$

$$\frac{8+9}{12} = \frac{17}{12} = 1\frac{5}{12}$$

FRACTIONS. . .
Multiplication and Division of a Whole Number by a Fraction. . .
Calculation and Rules. . .

. . .deducing the rule for dividing a whole number by a fraction:
According to the calculation utilizing the cross, as shown: In order to divide a whole number by a fraction, it is necessary to multiply the numerator of the dividend by the denominator of the divisor to obtain the numerator of the quotient; and to multiply the denominator of the dividend times the numerator of divisor to obtain the denominator of the quotient.

A SIMPLER RULE: In order to divide a whole number (or a fraction) by another fraction, it is enough to multiply the dividend by the inverse of the divisor.

Presentation #1: Multiplication of a Fraction by a Fraction

1. Present the operation on a slip: 1/3 X 1/3 = . The child shows the operation with symbols and operation signs. Then he must substitute ninths for the third in order to take one-third of the third. His result is 1/9.

1. If to multiply means "to take" then it means that I must take this 1/ 3 one-third time.
In order to do that, I must divide the 1/3 into three equal parts.
What is the equivalent fraction of 1/3 which is divided into three equal parts? NINTHS.
NOW I can take one-third of the 1/3.

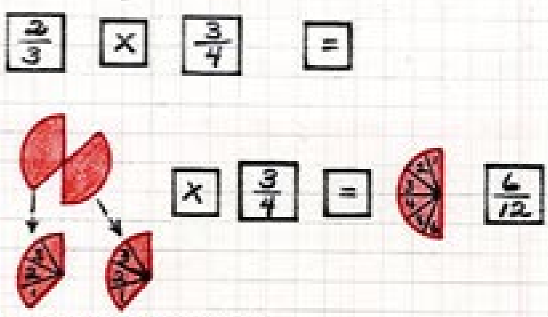
2. 2/3 X 2/3 = 4/9
The child must find the equivalent fraction which will show each of the thirds divided into three parts. Then he can take two-thirds of each of the thirds--- or 2/9.

2. Here we must take two-thirds of each of these 2/3.
So each of the thirds must be divided into three equal parts. . .then we can take 2.
What is the appropriate equivalent fraction? NINTHS.
And from each of the thirds we take 2/9. The produce, then, is 4/9.

3. 1/2 X 2/4 = 2/8
Here the equivalent fraction must be used which will divide the half into four equal parts: then the child can take two fourths.

3. We want to take two fourths of the half. . .so we must find the equivalent fraction which divides the half into four equal parts.
Then we take two of those fourths: 2/ 8.

4. 2/3 X 3/4 = 6/12



4. Here each of the thirds must be divided into 4 equal parts (twelfths) so that we may take 3/4 of each of the 2/3. OR 6/12. (Three twelfths from each third.)

5. CIRCLE CHART #17

Presentation #2: Passage to Abstraction: When the child has worked well with the material. . .

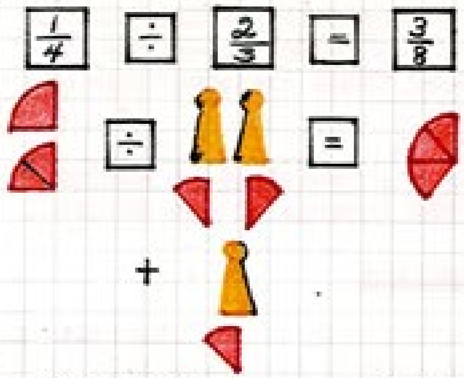
. . .we repeat the operation with the material and show the calculation which represents that operation.

Then we can deduce the rule: To multiply a fraction by a fraction, it is enough to multiply the numerator by the numerator; and the denominator by the denominator; then reduce the terms if necessary.

FRACTIONS. . .

Presentation #3: Division of a Fraction by a Fraction: Distributive Division

1. 1/4 ÷ 2/3 =



1. It is not possible to divide 1/4 between two thirds, so we must divide the fourth into 2 equal parts by finding the equivalent fraction. . .two eighths. WE WRITE THE NEW FRACTION ON THE BACK OF THE SLIP 1/4. Then one eighth goes to each of the two thirds. BUT BECAUSE THE RESULT OF MY DIVISION MUST BE WHAT ONE UNIT GETS, One eighth must also go to the third third. And the result is 3/8.

2. 3/5 ÷ 2/4 =

CIRCLE CHART #20.

Here we must find the equivalent fraction for the fifths which can be divided between the two fourths. Each of the fifths can be divided into two tenths---then each of the two fourths will receive three tenths. (Here we have discovered a denominator which will give a numerator which can be divided by 2---that is, it is the 6/10 which can be evenly divided; or we can see it as simply a matter of dividing each of the fifths into two parts for distribution.) THEN when each of the fourths has received 3/10, we must also give the other two fourths 3/10---and the quotient is 12/10.

OR. . .

An interesting variation: Instead of immediately finding the equivalent fractions for each fifth, we begin by distributing one-fifth to each of the two fourths. Then we have one-fifth which cannot be distributed equally, so it must be transformed into two tenths before distribution. At that point, the 1/5 + 1/10 might be transformed into 3/10...OR...1/5 and 1/10 can be distributed to each of the other two fourths. THEN WE ARE ADDING TOGETHER (1/5 + 1/10) + (1/5 + 1/10) + (1/5 + 1/10) + (1/5 + 1/10) = 4/5 + 4/10 = 4/5 + 2/5 = 6/5. The variation is interesting because it follows the pattern of distribution used with the decimal system materials. . .and also it gives us a result which does not reduce.

Presentation #4: Group Division: 2/3 ÷ 2/9 =

The problem here is stated in a particular way: How many groups of 2/9 can I form with 2/3. In order to answer that question, the dividend 2/3 must be transformed into an equivalent fraction made up of ninths. We change 2/3 into 6/9.

Then we take those 6/9 and discover how many groups of 2/9 can be formed. 3 groups.

2/3 = 1/3 + 1/3 = 3/9 + 3/9 = 6/9 divided into groups of 2/9 gives 3 GROUPS.

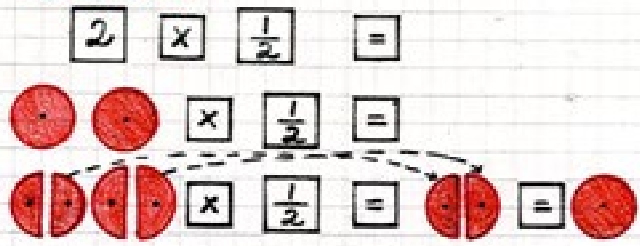
CIRCLE CHART #21

FRACTIONS. . .

Multiplication and Division of a Whole Number by a Fraction. . .

3. $2 \times \frac{1}{2} =$
The symbols and operation signs are placed on the mat; the quantity shown with the whole circles; each whole divided into two halves; then one-half of each whole is taken to show the product.

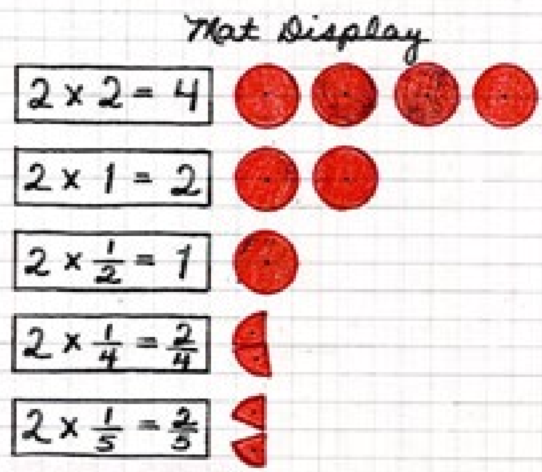
2. . . .means to take two wholes divided in half so that we can TAKE $\frac{1}{2}$ of each of these two wholes.



4. 2 X 1/4 =
In this operation, shown with the symbols and materials as in the preceeding, the wholes must be divided into fourths in order to take 1/4 of each whole. We take 1/4 of 2 wholes.

5. 2 X 1/5 =
Now the wholes must be divided into fifths so that we can take 1/5 of each.

6. Observe with the child: In the first two cases the answer is larger than the terms. In the third case it is smaller than the term we are multiplying.
CONCLUSION: The smaller the fraction by which we multiply, the smaller the product.



7. CIRCLE CHARTS 14, 15, and 16.

Presentation #2: Division of a Whole Number by a Fraction

Materials

1. The green unit stamps.
2. The small green skittles (from the stamp games or from the division memorization materials).
3. The cardboard fractions---circle insets.
4. The BIG SKITTLES FOR THE FRACTIONS: One black whole
One black divided into halves: red inside
One black divided into thirds: yellow inside
One black divided into fourths: green inside.

1. Preparation: $6 \div 3 = 2$.
The child shows the division with slips and symbols. Then he sets out 6 green stamps to show the dividend and three green skittles---he distributes the stamps.

1. The result of the division is what one unit gets.

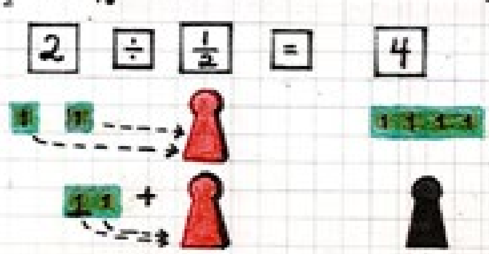
2. Preparation: $2 \div 2 = 1$. He makes this second division in the same way.

The quotient is what one unit receives.

3. Passage to the work: $2 \div 1 = 2$.
The child now shows only one skittle. A GOOD TIME TO INTRODUCE THE ONE WHOLE BLACK SKITTLE IN THE PLACE OF THE ONE GREEN.

3. Because there is only one skittle, the dividend is given to that one.
And the result is again what one unit receive[s]
In this case there is only one unit, and he has received 2.

4. $2 \div \frac{1}{2} = 4$.



4. Now we want to divide 2 by 1/2.
So we must use this skittle that is divided into two halves.
But our divisor is only 1/2. . .so we use only one half.
And to this one half I give the 2. . .because that is the quantity I possess to be divided.
BUT my result is what one WHOLE UNIT GETS.
So if 1/2 receives 2, then the other 1/2 receives 2 also. AND MY ANSWER IS 4.

Division of a Whole Number by a Fraction. . .

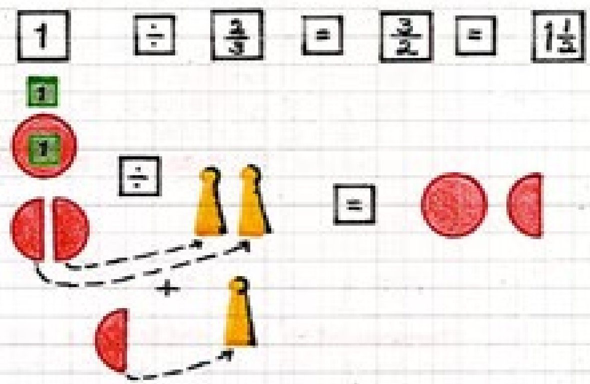
5. $2 \div 2/2 = 2.$
The child shows two green unit stamps, and divides them between the two skittle halves. Each half receives one unit. Then the two halves must be put together to discover what one unit receives: 2.

5. I have 2 whole units which must be divided among two halves.
So each half receives one.
But the two halves together make up one unit. . .and my answer is what one unit receives. So the answer is 2.

6. $1 \div 2/2 = 1$
Here the child must divide one whole unit between <u>two</u> halves. So he must divide the unit into two parts: he substitutes two red circle halves for the unit, showing first that the circle is the equivalent of the green unit stamp. Then each half receives 1/2; and the unit receives 2/2 or 1.

7. $1 \div 2/3 = 3/2 = 1\frac{1}{2}$
Here the skittle divided into thirds is introduced. And the one whole must be divided between two of the thirds. So it must be divided into halves: substitute the cardboard halves. Then each third receives one half. But the answer must be what 3/3 or one whole unit receives. . .so the equal quantity for the third third must be added to give the quotient. We cannot consider 1/3 of a child---we must see what the whole gets.

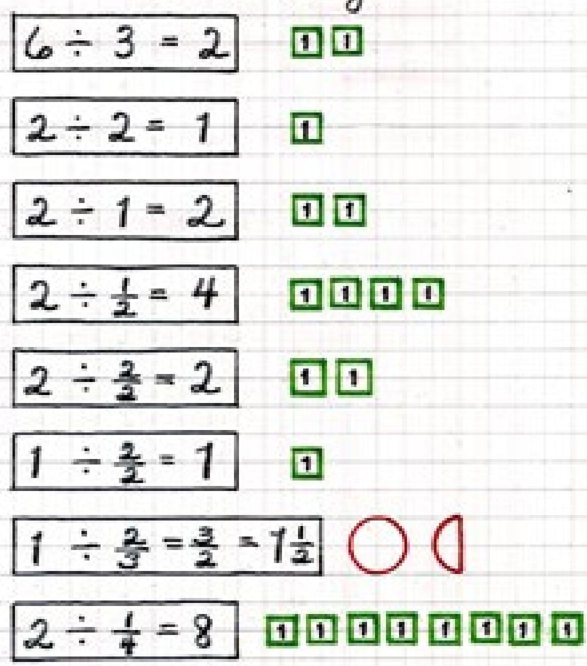
7.



8. $2 \div 1/4 = 8$
Now the skittle divided into fourths must be used. And the entire quantity of 2 is given to the one fourth. But to show what one unit receives, each of the other three fourths must also receive 2. Therefore the answer is 2 X 4 or 8.

9. CIRCLE CHARTS 18 and 19.

Mat Display



Presentation #3: Writing the Operations; Giving the Rules

When the child has worked with the materials for a time, we ask him to show a particular operation (in multiplication or division); and then, from the observation of that operation and its result, we discover with a calculation what happens:

MULTIPLICATION: $2 \times \frac{1}{5} = \frac{?}{?} = \frac{2}{5}$ $\quad 2 \times \frac{1}{5} = \frac{2 \times 1}{5} = \frac{2}{5}$

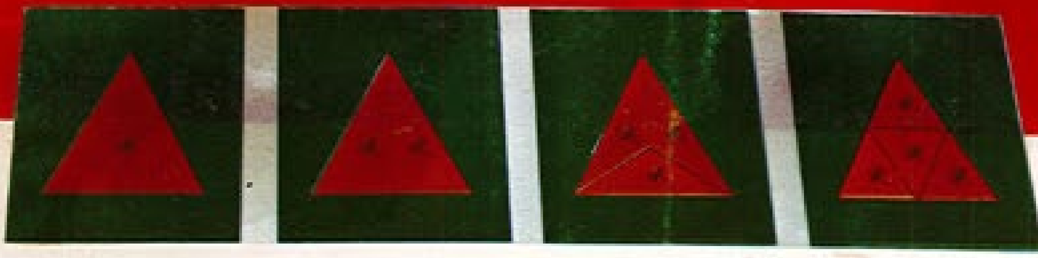
$2 \times \frac{3}{4} = \frac{2 \times 3}{4} = \frac{6}{4}$ $\quad 2 \times \frac{1}{4} = \frac{2 \times 1}{4} = \frac{2}{4}$

THEN, from a series of such calculations, we are able to deduce the rule:
In order to multiply a whole number times a fraction, it is enough to multiply the numerator times the whole and the denominator stays the same.

DIVISION: $2 \div \frac{1}{2} = \frac{?}{?} = 4$ $\quad \frac{2}{1} \div \frac{1}{2} = \frac{2 \times 2}{1 \times 1} = 4$

Here we must explain that if we want to write a whole number as a fraction, we use the whole number as the numerator and the denominator is 1.

We can show the cross. . . or turn the second fraction, the divisor, upside down!



FRACTIONS: An Introduction

Material

1. The series of metal circle insets---the real fraction material, chosen because the circle is easy to divide into many equal parts or a few. . .but can be divided even into the smallest fraction. This set includes the whole, halves, thirds, fours, fifths, sixths, sevenths, eighths, nineths, and tenths.
2. The series of square metal insets---divided into whole, halves, fourths, eights, and sixteenths---all shown as obtained by the diagonals (which gives triangles) and as obtained by joining the midpoints, giving squares and rectangles.
3. The series of triangle metal insets, with the whole, halves, thirds and fourths. (above.)

NOTE: All of these metal fractional insets are red framed in green for easy recognition.

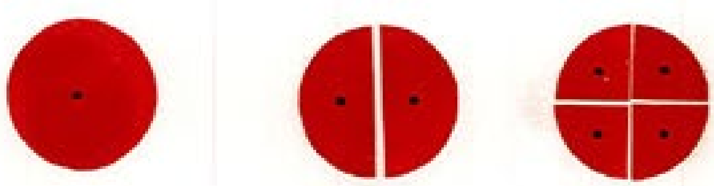
4. A bowl or box containing little white tickets which label all the fractional parts of the triangle, square, and circle. There is one for each part of each inset in each series.
5. A golden bead, an apple, a knife.

Presentation: Group

1. Ask the children: The fraction material is displayed.	1. Where does numeration start? Where do we start counting?
2. Show the golden bead.	2. One is represented by this golden bead. Up until now, it has been indivisible. We have had the impression that this bead is the beginning of all numbers. In fact, to this bead we can add many many more numbers to infinity---but we cannot take 1 away from it. If I want to divide it, I must take a hammer and smash it up.
3. With the inset as a guide, draw a whole circle inset on paper. Cut it out and place the circle in the frame---on it place the top of the paper circle.	3. If I could take this bead and break it up into very small bits, it would become something round flat like this.

FRACTIONS. . .
Presentation. . .

4. Show how an apple can be cut into parts.	4. With this flat circle of paper, I could cut it into equal parts. I couldn't do that with a hammer. We could also cut an apple into equal parts. If I have an apple and I want to divide it between two children, what must I do? I will cut it and give each one half. Now I have no longer one unit, like the bead. But I have two halves. Suppose we had four children. I could cut each half again to make four parts. And what part of the apple would one child receive? One quarter---one fourth. The important thing is to cut the apple carefully to make four equal parts. We have shown, then, that although the bead is too hard to be divided, we CAN DIVIDE ONE UNIT.
5. Cut the paper circle and cut it unevenly. Display the two parts on the mat.	5. We can cut this paper circle in half, too. Did we cut it into two equal parts?
6. Display on the mat now the whole circle inset, the halves and the fourths.	6. This has been divided into two perfectly equal parts---like an apple. This whole has been divided into 4 parts. Each part is one fourth. Of course, they must be divided
First display separately--- then fit together for the whole.	exactly into equal parts. See how exactly they form the whole.



FRACTIONS. . .
Presentation. . .

7. Introduce the term fractions.

7. What do we call these parts of a number?
We have been saying one-half, one-fourth.
These parts of a number are called fractions.
Fraction comes from the Latin word which means "to divide."
In order to obtain fractions, it is necessary to divide the unit.

8. Discuss the use of fractions.

8. Do you think in our daily lives, that we use only whole numbers or fractions?
Let's see if that is true.
At what time do you come to school?
8:30: 8 is a whole number, but 30 tells us "one half" an hour.
When we buy milk, sometimes we buy $\frac{1}{2}$ quart or $\frac{1}{2}$ pint.
When we buy butter, we buy $\frac{1}{2}$ pound.
We DO use fractional numbers.
THE FRACTIONAL NUMBERS ARE SMALLER PARTS OF A WHOLE UNIT.

9. Introduce the materials.

9. Now that we have understood what fractional numbers are, and that a whole number can be divided, we want to know out material better.
Let's see what these are:
What is this? A circle.
What is this? A square.
What is this? A triangle.
This is one whole square. I can call it a unit.
This is one whole circle. . .
All three correspond exactly to one bead.
Let's see how I can divide them.

10. Show all the halves. . .laid out together on the mat.

10. Let's see how the circle is divided.
Here it is divided into 2 parts.
And here is the square divided into 2 parts.
This is the square divided into 2 parts another way.
Here is the triangle divided into 2 parts.
Can you tell me what part we have when we have divided a whole into 2 parts?
This is a half, and this is a half. . . . .

11. Leaving the halves displayed on the mat to one side, proceed to the thirds, laying them out side by side.

11. Here the circle has been divided into three parts.
And the triangle has been divided into three parts.
Each of these parts is one third.

12. Proceed with the demonstration of the materials, showing the fractions in groups through the tenths. Be sure the child knows that a whole can be divided into equal parts to obtain the fractional numbers. And that he knows what the names for those fractional parts are.

Presentation: An introduction to the writing of fractions.

1. Discuss the families of fractions, as a preparation for writing them.

The fractional materials are all displayed XXXXXXXXXXXX grouped together as introduced in the first presentation.

OR

the insets may all be replaced in the frames and pointed out as they are mentioned.

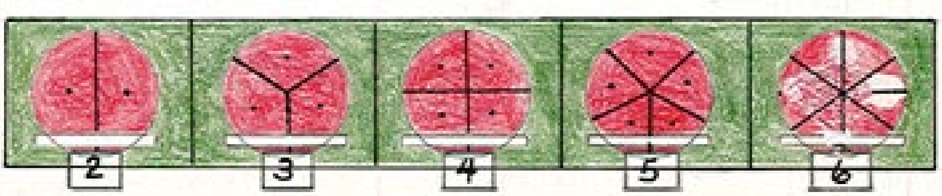
Show the square of 16th if not previously presented.

1. Now we want to know how fractions are written.
Tell me your name.
Your last name.
Your father's last name, your mother's, your brother's.
Then---Smith is your family name.
Now, what is your name?
What is your father's name?
So---each member of your family has two names: their own name and the family name.

All of these fractions have a family name.
Each family has its special name.
This is the family of halves.
This is the family of thirds.
How many members are in the family of halves? 2
How many members in the family of thirds? 3
Here is a very numerous family.
This is the family of 16ths.
Count how many members it has.

2. Place thin white paper strips on the face of each inset frame below the figure---the figures should have been replaced in the frames during the discussion.

2. We are going to write the family name below this white line.
This is the family of halves---it has been divided into two parts.
We'll show its family name with this 2.



FRACTIONS. . .

Presentation: Writing Fractions. . .

3. Explain the numerator name and function.

3. Now we know the last names of all the families.
But if I want to address one member of a family, I cannot say "Smith" calling them by their last name.
So if I want only 2 members of the 7th family, I can't just write seventh---that means everyone.
So I name which members I want and I write that number above the line.
This number tells me how many members of one family I want.

Use another white square of paper, write a 2 on it and place it above the line on the fraction named.

4. Now move the two sevenths to the mat---and below it place

$$\frac{2}{7}$$

4. This tells me that I have two members of the seventh family.

5. Write another slip: 3.
Put it in place of the 2 and add another seventh to the display.

5. Here I have three sevenths.
It is still the seventh family, but instead of 2, I am showing 3.

State the rule.

Therefore, the number below the line tells me the family's name. And the number above tells me how many I want.

6. Distribute the small prepared tickets to the children. There is one for every fraction. The children then label each fraction, laying the small cards on each part.

7. Now the children are ready to work by themselves.
   a) They can take the parts they want from one family, trace those parts in their notebook, color them red, and write down the fraction: $\frac{1}{2}$, 2/4, 3/7.

   b) They can cut out the fractions from paper, write the fractional number.

They should work with these activities for a long time---2 or 3 months. When they have mastered all the fractions, pass to the last part: the name of the numerator and the denominator.

Presentation: The names numerator, denominator, fraction---and what they are.

1. Tell the children:

1. Why do we call the parts of a unit a fraction?
What do we really do to the whole?
What do we do to a number to make fractions? We divide it.
Then---what is a fraction?