THE ALTITUDE OF THE TRIANGLE. . .
The Orthocenter. . .

EXERCISE: We invite the child to take again into consideration those contour drawings he made during his exploration of the altitude of the triangle with the constructive triangles. In each of these seven triangles of reality, the child has already identified the three altitudes in red. Now we ask him to identify the orthocenter of each triangle, extending those altitudes of the obtuse-angled triangles if he has not done so to show the orthocenter.

NOTE: Give the etymology of orthocenter as he begins this work; from the Greek orthos meaning straight and center which, unlike the center of the circle, here means the point of concurrency---the meeting point. And so the orthocenter is that point where the straight things, which are the altitudes (perpendicular lines by definition) meet.

IN CONCLUSION: There are three points of concurrency.
Three RESEARCHES of discovery:

A. Using a blue color, determine the point of concurrency of the medians in each of the seven triangles of reality. Give the name of this point of concurrency.
B. Using a green color, determine the point of concurrency of the axis (the penpendicular bisector of each side) in each of the seven triangles of reality. Give the name of the point.
C. Using yellow color, determine the point of concurrency of the angle bisectors in each of the seven triangles of reality. Give the name of this point.
D. Using the color red, blue and green, determine the points of concurrency of the altitudes, medians and axis respectively in each of the seven points of concurrency of each triangle.

SPECIFIC NOMENCLATURE OF THE RIGHT-ANGLED TRIANGLE

Presentation

1. Ask the child to construct the two right-angled triangles of reality. Identify them.

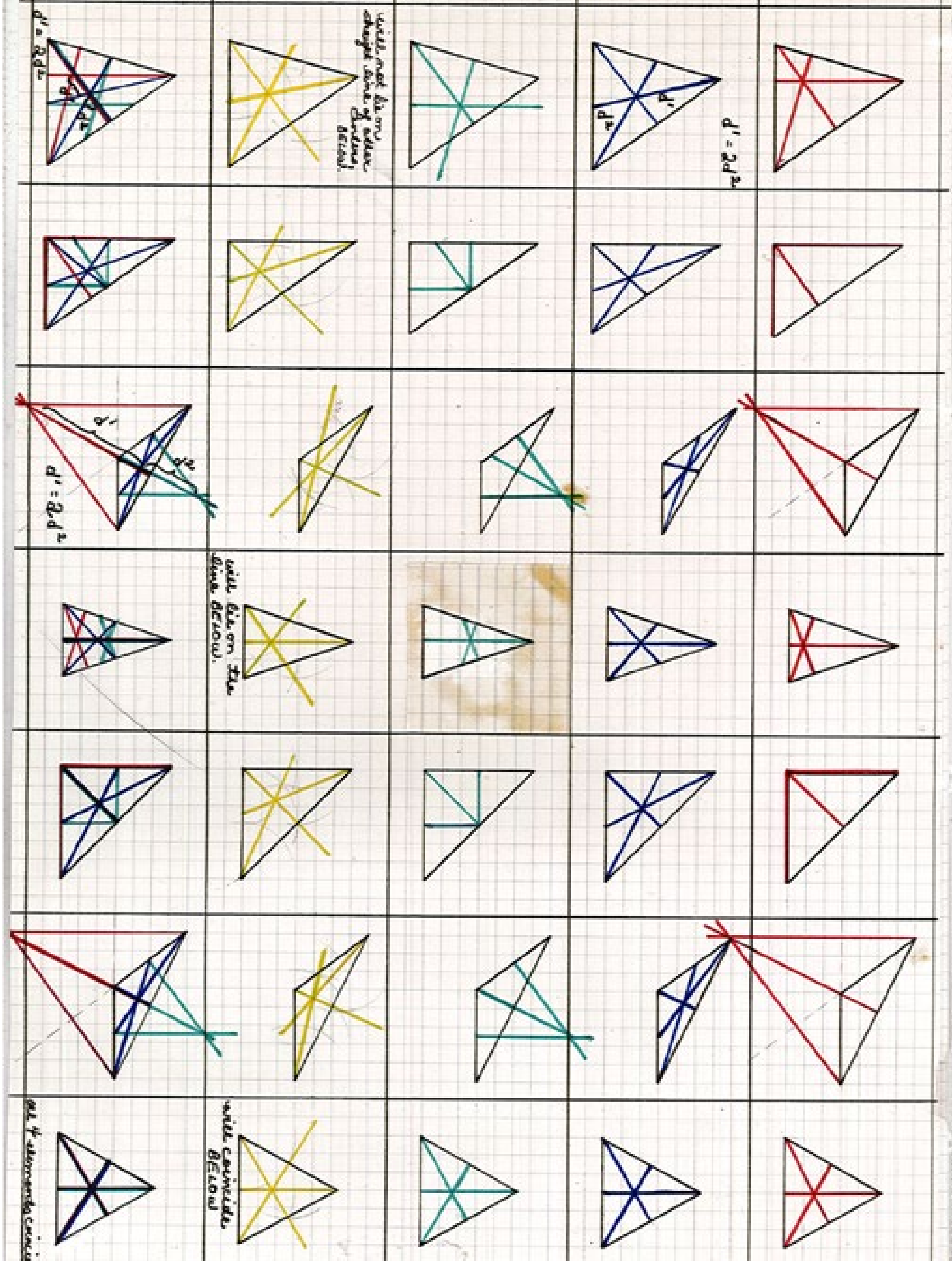
1. We have the right-angled scalene triangle, the triangle which shows the Pythagorean triple. (sticks 12, 16, 20)
And we have the right-angled isosceles triangle constructed with two equal sticks and a third from the special neutral sticks for the construction of this right-angled triangle.

2. Review the nomenclature of the two triangles with which the child is familiar: the triangular region, the sides, the vertices, the altitudes, the axes, the medians, the bases, the angles. And CONCLUDE: this nomenclature can be applied to all triangles, regardless of the kind of triangle.

3. Introduce the nomenclature of the right-angled triangle.
Show the measuring angle within the triangles.

3. The names of the sides of the right-angled triangle are special.
The two sides which constitute the sides corresponding to the measuring angle --- that is, the sides of the right angle--- are called catheti (a Greek word) or legs.
In the isosceles triangle, the legs are equal.
In the scalene triangle, the legs are not equal; so we have both a major leg (the longer of the two), and a minor leg.
The third side of the right-angled triangle is called the hypotenuse.

Altitude orthocenter median centroid Axis circumcenter = Aeuler's Line	Angle Bisector incenter	Axis circumcenter	Median centroid	Altitude orthocenter



THE STUDY OF THE NOMENCLATURE. . .

Now we begin examining the details of the polygons; and the first of those studies is the study of the TRIANGLE. It is divided into three parts:

I. Triangles according to the sides.
II. Triangles according to the angles.
III. Triangle analysis according to the sides and the angles.

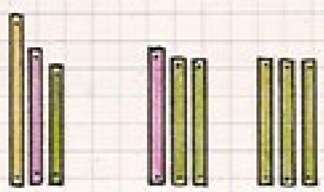
Part I: The Study of the Triangle according to the sides.

Presentation

1. Ask the child for three sticks of unequal length. . .three different colors.

2. Then ask for a second group of three: this group composed of two of the same length (same color), being one of those lengths used in the first group of three unequal sticks and the third being also one of the lengths used in the first group.

3. Then ask for a third group of three sticks composed of three equal sticks, the same length (same color) as the two equal sticks in the second group.



OR

to the same end,
Request first three equal sticks; then a group of three with two equal to the three in the first group; and finally a third group of three of which one is the same length as those two equal in the second group, one is the same length as the third in the second group and one other stick.

4. Ask the child to construct triangles with each group of three with paper fasteners. Display those three triangles.

5. Take the triangles one at a time and identify them, giving the nomenclature and the characteristics.
NOTE: the child may know the names at this time, but this is an important moment for positive identification and knowledge of the three kinds of triangles classified by the sides.

5. This is a scalene triangle because it has three sides of different lengths.
This is an isosceles triangle. It has two equal sides and one different one.
This is an equilateral triangle. It has three sides of the same length: three equal sides.

6. Review the etymology of the three triangles.

7. Give a vivacious three period lesson.

Part II: The Study of the Triangle according to the angles.

Material: to the material already being used, add the envelope which contains a bright yellow scalene triangle, the right angle being clearly marked in red. This is the measuring angle. Its main characteristic is that it looks the same from both sides. Note the red color of the angle, our point of attention.

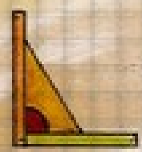
Presentation

1. Introduce the triangle with the measuring angle.

1. This triangle will help us with our discoveries about triangles.
We can see that it has one right angle that is colored red.
This is our measuring angle. We will use it to identify other angles of our triangles.

The Study of the Triangle According to the Angles. . .
Presentation. . .

2. Ask the child to form triangles with the sticks which have one right angle. His guide is that he should use three different lengths of sticks each time, and that he must use the measuring angle to be certain that two of the sticks form an angle of 90°.



Is there a third stick that will now close this triangle?

2. Now we want to construct triangles according to their angles.
We begin by choosing two sticks of different lengths.
Then, using the measuring angle, we make certain that these two form a right angle.
Finally, we take a stick of a third length and fit it between the first two.
It must fit without changing that first right angle we form.
In looking for the third stick which will close the triangle, we may have to try different lengths. . .and sometimes we will not be able to find one at all.

3. The child makes the exploration until he discovers those two triangles which we can construct with the sticks, as noted in the discussion of the Pythagorean triples. (6, 8, and 10 sticks: orange, red and black;
12, 16, and 20 sticks: beige, pink and yellow)

NOTE: IF THE CHILD DOES NOT DISCOVER the two triangles, we show them.

In order to construct a right-angled triangle, we must have a special combination of sticks.
We can use these three: the orange stick, the red stick and the black stick.
Let's begin our construction with the two shorter sticks.
Together they form an angle.
Now we slide the measuring angle down the vertical stick, the red stick; and we close the yellow stick at the bottom so that it forms a right angle, shown by the measuring angle.
Now we can take this third stick, the black one, to close the triangle.
There is one other combination of sticks with which we can construct the right-angled triangle: the beige, the pink and the yellow
Would you like to construct that right-angled triangle?

Display only one of the two triangles here constructed for the following part of the presentation.

4. Ask the child to unite two more sticks. Then to show a right angle with the measuring angle between them. Then ask him to form an acute angle before the third stick is chosen to close the triangle.

4. Now I want to form an angle smaller than the measuring angle---an acute angle.
Now we close the triangle with a third stick.

5. Measure each of the three angles when the third stick has been placed, showing that each of the angles is acute.

Now let's examine all three of the angles in the triangle we have constructed.
This is an acute angle. . .and this one. . . and this one.
All three angles of this triangle are smaller than the measuring angle. . .
all three are acute angles.

6. The child now constructs an obtuse-angled triangle, laying out first the longer of the two sticks he chooses, placing the measuring angle and bringing the shorter stick away from that angle to form an angle larger than the right angle.

6. Now we want to construct an obtuse-angled triangle.
With the first two sticks we choose we will form an angle greater than the right angle.
Then we have met our specification for an obtuse angle and we close the triangle with a third stick.

7. Begin a second analysis of the triangles by their angles with the right-angled triangle. . .showing that it has one right angle. . .then the acute and the obtuse-angled triangles are examined according to angles.

7. We are going to consider the three triangles we have constructed again; and analyze them.
What triangle is this?
How many right angles does it have?
We must measure the other two----what are they?
Therefore, we have a right-angled triangle because we have one right angle and two acute angles.

Measure the angles of the a cute-angled triangle.
Verify the obtuse angle with the measuring angle.

This is an acute-angled triangle because all its angles are acute.
This is an obtuse-angled triangle because it has one obtuse-angle.

8. Show that the right-angled triangle can have only one right angle: take the sticks apart at one vertex, show the original right angle between two sticks and try to construct another between the next two.

8. Is it possible for the right-angled triangle to have more than one right angle?
It is evident that this triangle cannot be closed if I have more than one right angle.
Therefore the right-angled triangle cannot have more than one right angle.
And what are the other two angles? acute.
This triangle takes its name from the one right angle

9. Note that the acute-angled triangle has three acute angles.

9. Let's measure the angles of the acute-angled triangle again.
How many acute angles does it have?

10. Show that the obtuse-angled triangle cannot have more than one obtuse angle. Take apart the sticks at one vertex---try to make a second obtuse angle and close the triangle.

10. Is it possible to make another obtuse angle here in this obtuse-angled triangle?
If I have two obtuse angles, it is obvious that I cannot close this triangle with only one more stick.
So the obtuse-angled triangle has only one obtuse angle.
What are the other two? acute

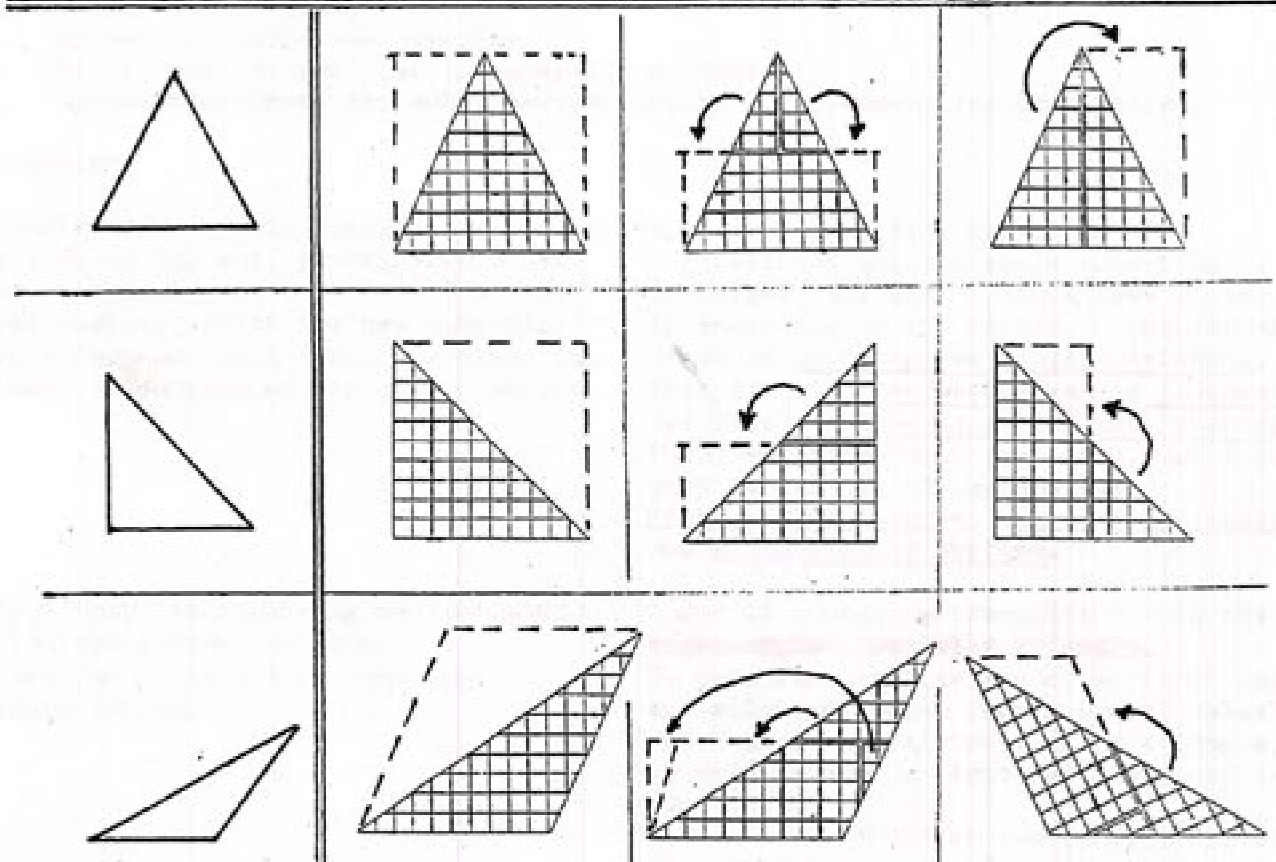
11. Compare the number of acute angles in each.

11. How many acute angles does this right-angled triangle have? the acute-angled triangle? the obtuse-angled triangle?

FINALE: Now show all six of the triangles constructed to this point---the three by sides, the three by angles. Take the triangle drawer from the geometry cabinet and match these six figures with the six on the plane. Show these latter six in the same arrangement as these in the drawer.

We have constructed six triangles.
Look at the six triangles found here in the cabinet drawer.
Can we match each of these to the six we have constructed?
On the first row we have: the scalene, the isosceles and the equilateral triangle.
On the second row we have: the right-angled, the acute-angled and the obtuse-angled triangles.

	double the area	bisect the height	bisect the base



	$A = \frac{bh}{2}$	$A = b\frac{h}{2}$	$A = \frac{b}{2}h$

The Study of the Triangle According to the Sides and the Angles. . .

6. Take the triangle drawer, remove the inset figures, classify them according to both sides and angles---and match each to the corresponding constructed triangles.
In the drawer: right-angled scalene,
acute-angled isosceles,
acute-angled equilateral;
right-angled isosceles,
acute-angled isosceles,
obtuse-angled isosceles.

6. In the drawer I have six plane triangles. Let's classify each of these six according to both their sides and their angles.
Now can we match these triangles to the ones we have constructed.
We have constructed the right-angled scalen
and the acute-angled isosceles,
the acute-angled equilateral,
the right-angled isosceles,
Here is a second acute-angled isosceles.
And we have no obtuse-angled isosceles here constructed.

7. Construct an obtuse-angled isosceles with the sticks. Show that the obtuse-angled isosceles in the drawer has two equal sides by turning it backwards in the frame. MATCH THE TWO.

NOTE: this ommision may be that of the acute-angled isosceles: in that case, we construct it here instead.

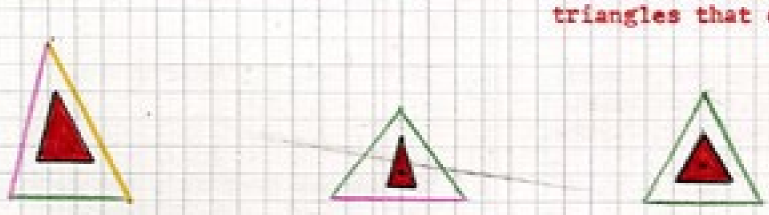
7. We must construct an obtuse-angled isosceles triangle.
We should take two short sticks of the same color.
Then we must measure with the measuring angle a right angle between them---and then increase that angle to make it obtuse.
Now we close it with a third long stick .

8. Analyze the result of the match. There are two constructed figures unmatched: the acute-angled scalene and the obtuse angled scalene. Use the additional insets for these two. MATCH.
Two insets (acute-angled isosceles) have been matched to one constructed triangle. Put aside the one less easily recognized: that is, the one from the top row of the cabinet drawer.
Two of the constructed triangles are acute-angled scalene. Eliminate one of these two to show only ONE TO ONE.

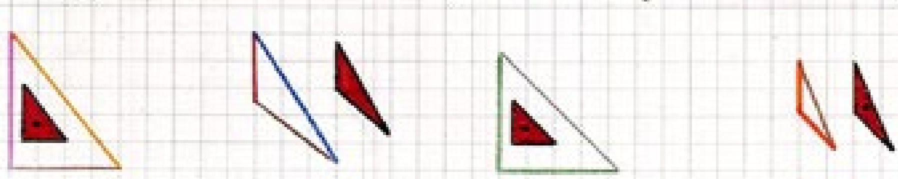
Now Let's see what has happened.
We have two constructed triangles which we have not matched with insets.
In this box I can find these two triangles: the acute-angled scalene and the obtuse-angled scalene.
NOW WE HAVE two acute-angled scalenes that we have constructed---let's show only one of them with this new inset.
And we can eliminate one of these two insets that we have matched with this one constructed acute-angled isosceles.

9. Now we have shown seven triangles. Announce that these are THE SEVEN.

9. Here we have only seven triangles.
In reality these are the only seven triangles that can be formed.



acute-angled scalene triangle — acute-angled isosceles triangle — acute-angled equilateral triangle



right-angled scalene triangle — obtuse-angled scalene triangle — right-angled isosceles triangle — obtuse-angled isosceles triangle

10. Replace the inset figures in the cabinet drawer. Take out the center frame from the top row and replace with the card-frame of the acute-angled scalene triangle and place on top of it the cardboard inset. To the side show the card-frame of the obtuse-angled scalene and the corresponding inset piece.

11. Review the nomenclature of each as it is replaced and note the correspondence again to each of the seven constructed triangles.

12. Conclusion statement.

12. These are the only seven triangles. We cannot form another triangle that does not have the characteristics of one of these.

EXERCISE: The child constructs with the sticks those triangles he wishes. He classifies each according to sides and angles. He may duplicate his work, but he will find that he cannot make more than 7 different ones.

NOTE: To say that three sticks make a triangle is not enough. There must be a certain proportion in the length of the sticks. The third side must be smaller than the sum of the other two. When that third is equal to the first two, we have the limit.

THE EQUILATERAL TRIANGLE

Material

1. From the triangle drawer, the acute-angled isosceles, the acute-angled equilateral, and the acute-angled scalene (an additional figure that has been placed in the drawer in the preceeding presentation.)
2. Slips of paper.

Presentation

1. Display the acute-angled isosceles triangle, the acute-angled equilateral triangle and the acute-angled scalene triangle on the mat.

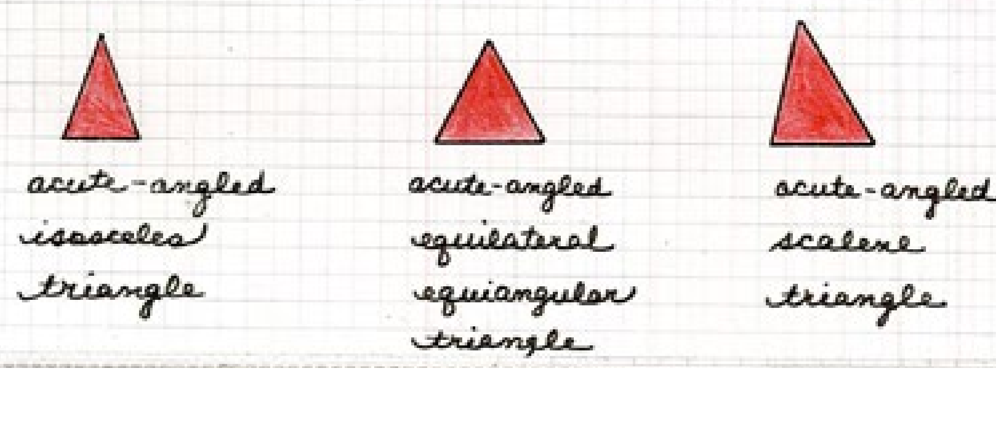
2. Ask the child to identify each according to sides and angles and write a slip to label each one.

3. Tear the slips into three pieces: acute-angled/isosceles/triangle. Put in a column below each figure.

3. Let's look at these words.

4. Take the equilateral and replace it in the frame; rotate it to show that each of the angles is equal. Then add a slip below the triangle that reads equiangular.

4. We can see that this triangle is both equilateral and equiangular. So we can add the word equiangular to describe this triangle. We have two qualities for the first and the third triangle. Now we have found one more for the equilateral triangle. We have 3.



5. Turn over the paper that reads "isosceles" under the first triangle, leaving only the identifying word "acute." Show that the one word is not enough to indicate the above triangle. Then turn over "scalene" to again show this. Then turn over "acute-angled" on each of these two, showing again that both words are needed to define both triangles.

NOTE: We may point out during this game that the equilateral triangle is an isosceles triangle.

5. Let's play a game.
We'll remove one quality from below one of these triangles and see if we can still identify that triangle.
If we take away "isosceles" from the first triangle, what do we read?
"Acute-angled triangle."
Is that enough to identify this triangle?
No.
What if we take "acute-angled" from the last triangle?
It is not enough to identify this as a "scalene triangle." I could take a right-angled scalene or an obtuse-angled scalene.
AND SO I need both of these identifying qualities to describe both of these triangle

6. Go to the descriptive labels below the equilateral triangle, showing that it can be described with only one quality: equiangular or equilateral.

6. Now let's take away one of the words which describes the equilateral triangle.
Can we take away the "acute-angled" and still identify this triangle? YES
Therefore we know that acute-angled is not a fundamental word for describing the equilateral triangle.
Now let's turn over the word "equilateral", too.
Can we describe the triangle with only the word "equiangular?" YES
Can we describe it with only the word "equilateral?" YES
Is it enough to describe it with the word "acute-angled?" NO

7. Conclude with a definitive statement about the equilateral triangle.

7. To describe this triangle, one characteristic is enough.
We may describe it with the word which establishes its sides or its angles--- the equality of each.
When we say equilateral, we imply equiangular and acute-angled. When we say equilateral, we know the other two.
Both this first and this third triangle need two characteristics to adequately describe them; but the second triangle needs only one---THE RIGHT ONE.

NOTE: An important detail of new math is here introduced. We approach the real definition of the isosceles triangle which is that it has at least two equal sides. That is, the equilateral triangle is an isosceles triangle with three equal sides, but it is an isosceles. . .and more. The opposite is not true: all equilateral triangles are isosceles triangles, but all isosceles triangles are NOT equilateral triangles.
As the child works with the construction of triangles, he knows he must have then AT LEAST TWO EQUAL STICKS to form the isosceles triangle.

THE CONSTRUCTION OF THE TRIANGLES

We know, at this point, that there are only seven triangles in reality. It may happen that, when the child works by himself constructing scalene or isosceles triangles, he may choose sticks which meet the specifications for those triangles that we have given him; but which will not, in fact, construct that triangle. It is time to see why; and under what circumstances this happens. We are investigating the Theorem: that is, the relations of inequality between the lengths of the sides of jointed triangles. (those formed with the sticks; when we draw a triangle, it isn't jointed.)

Presentation

1. In constructing a scalene triangle, if the child has chosen at random three sticks, the two shorter ones having a sum less than the longest stick, we present SCALENE Case #1.

   Show the sticks of construction and the constructed triangle at the side of the mat.

1. Let's examine the construction of the scalene triangle.
We know that we must choose three sticks of different lengths.
You have chosen a purple, a brown and a yellow. . .so we do have three different lengths.
FIRST let's unite each of the shorter sticks to the ends of the longer one.
Now let's try to unite these two ends.
The more we open them, the greater the distance between them.
THIS IS THE CASE OF IMPOSSIBILITY.
It is impossible because the sum of the minor legs is less than the longest---the third---side.

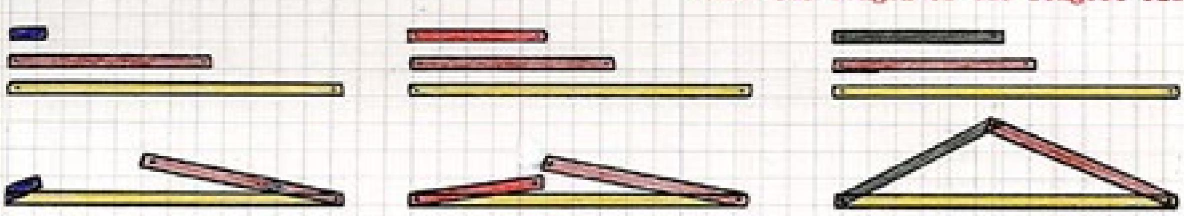
2. Examine SCALENE Case #2: keep two of the three lengths used in the presentation of the first case; use the third stick which, together with the first shorter stick, will equal the longest stick.

   Show the sticks of construction and the constructed triangle below the first case on the side of the mat.

2. Now let's try another construction.
We'll use this yellow stick again and the brown one, but let's try the red stick with these two.
First we unite the two shorter ones to the longest at each end.
Then we try to unite the two loose ends.
This time is doesn't work because the sum of the minor legs is equal to the length of the long side.
THIS IS THE LIMIT CASE.

3. Now construct the possible case, retaining again the first two sticks, but using the next possible length that will not give an isosceles.

3. Now let's use the yellow and brown stick once more, but let's use a slightly longer stick for the third side.
We know red is the limit for this triangle and we can't use another brown, for it will give us an isosceles triangle.
With the black or the green stick we find that it is possible to close the triangle.
We have three different sticks: short, medium and long.
THIS IS THE POSSIBLE CASE.
The sum of the two minor legs is longer than the length of the longest side.



NOTE: At the sensorial level, the child is discovering in his exploration the PRACTICE OF THE THEOREM.

AGE: 6

THE CONSTRUCTION OF THE TRIANGLES. . .

Presentation: ADVANCED LEVEL: Examining the Three Cases of Scalene Construction
For the child of 8, 9 years.

NOTE: Here we are constructing the same triangles as in the first presentation, but now we are considering the length of the stick. (yellow:20, brown:12, purple:4, red:8, black:10) This represents the distance between the holes of the sticks.

1. Invite the child to give the lengths of the sticks used in the construction of the Case #1 triangle.
   a) Introducing the "greater than" sign, show the notation of the case. CONCLUDE:
   b) Introduce the "less than" sign in the reverse notation.

1. The three lengths: 20, 12, 4.

$20 > (12+4)$
$20 > 16$

Now I know we can't construct this triangle because the longest side is longer than the other two sides.
Now let's consider the smaller sides first:

$(12+4) < 20$
$16 < 20$

The symbol that meant "greater than" becomes "less than" in this position.

2. From the triangle constructed for Case #2, show the notation quickly. It is not very interesting.

2. $20 = (12+8)$
$20 = 20$ or
$(12+8) = 20$
$20 = 20$

3. Moving from the triangle constructed for Case #3, show the notation and its inverse.

3. $20 < (12+10)$
$20 < 22$
and
$(12+10) > 20$
$22 > 20$

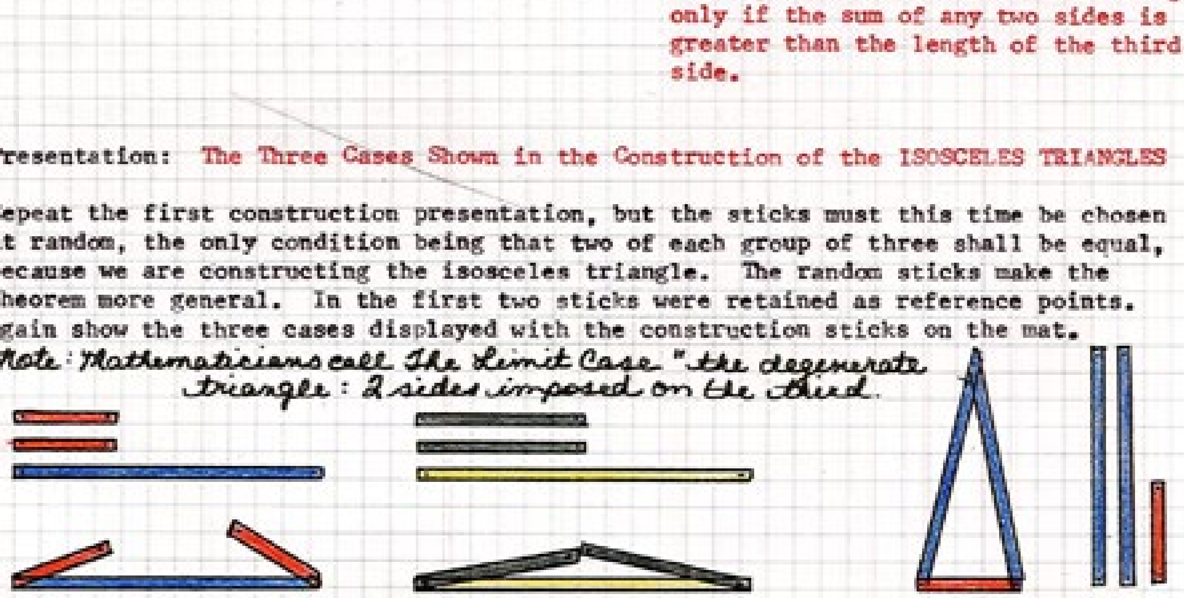
4. Organize the theorem:

4. It is possible to construct a triangle only if the sum of any two sticks is greater than the length of the third stick. That is,
It is possible to construct a triangle only if the sum of any two sides is greater than the length of the third side.

Presentation: The Three Cases Shown in the Construction of the ISOSCELES TRIANGLES

Repeat the first construction presentation, but the sticks must this time be chosen at random, the only condition being that two of each group of three shall be equal, because we are constructing the isosceles triangle. The random sticks make the theorem more general. In the first two sticks were retained as reference points. Again show the three cases displayed with the construction sticks on the mat.

Note: Mathematicians call the Limit Case "the degenerate triangle: 2 sides imposed on the third.



For Advanced Level

Case #1: Impossible
$18 > (6 \times 2)$
$18 > 12$
$(6 \times 2) < 18$
$12 < 18$

Case #2: The Limit
$20 = (10 \times 2)$
$20 = 20$
$(10 \times 2) = 20$
$20 = 20$

Case #3: Possible
$6 < (14 \times 2)$
$6 < 28$
$(14 \times 2) > 6$
$28 > 6$

THE STUDY OF THE TRIANGLE

Presentation: The Specific Nomenclature of the Triangle

Material: the same with the wooden triangle stand, and a plumb line.

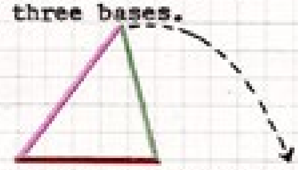
1. Ask the child to take any three sticks and construct a triangle. Give the familiar nomenclature and add the triangular region.

1. This is the triangular region.
This is the vertex of the triangle.
This is the side of the triangle.
This is the angle of the triangle.

2. Introduce base and altitude.

2. The triangle has no diagonals.
It has two new elements.
The triangle has a base and it has an altitude.
Really, THE TRIANGLE HAS AS MANY BASES AS IT HAS SIDES: AND AS MANY ALTITUDES AS IT HAS BASES.

3. Rotate the triangle, pointing out the three bases.



3. Now my base is the brown stick.
Now the green stick plays the role of the base.
Now the pink stick is the base.
I have three bases: as many bases as I have sides.

4. Repeat the nomenclature, adding the name of base.

4. What is this?
And what is this?
Show me the base.
It is called the base because the triangle rests on this side.

5. Introduce the concept of the altitude by asking three children of different heights to stand against the wall. Measure each with a chalk mark. Compare the heights: a good exercise in the adjective grammar box with comparative and superlatives.

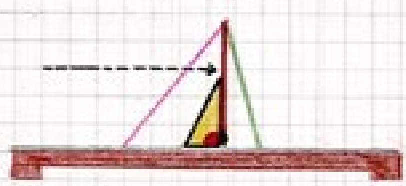
5. John is taller than Jean.
Jean is not as tall as John.
Peter is shorter than John and taller than Jean.

6. Rotating the t riangle now on the three bases, show the three heights of the triangle.

6. Let's see how tall this triangle is.
With this base, its feet are the green stick.
Now, with brown feet, this triangle is taller.
When its feet are pink, we have the shortest triangle.

7. Introduce the wooden triangle stand, a rectangular holder which has two metal strips along side a slit, allowing one to stand the triangle upright. Choose one base and fix the triangle between the metal holders.

7. Now this triangle stands up by itself.
Let's stand it on the green base.



8. Let the plumb line drop between the space in the holder, moving it from one end until it hangs from the top of the vertex. THE STRING IS RED to emphasize the altitude which we are showing.
Show the base angle with the measuring angle.

This is the altitude of this triangle according to this base.
WE CAN SAY THAT THE ALTITUDE IS PERPENDICULAR TO THE BASE.

THE ALTITUDE OF THE TRIANGLE. . .
Presentation: Nomenclature. . .

9. Rotate the triangle, showing another base in the stand. Repeat the experience of showing the altitude with the plumb line and measuring with the measuring angle, verifying the perpendicular lines at the base.

9. This is the altitude according to the green base. We see again that the altitude is perpendicular to the base.

10. Repeat a third time with the third base.

10. We can conclude that the triangle has three bases and three altitudes.

Presentation: ALTITUDES

Material
1. The wooden triangle stand with metal ridges.
2. The plumb line with red string.
3. All the triangles in the material we have used up to now:
   a) Those in the geometry cabinet.
   b) The constructive triangles, second box of first series: blue.
   c) The triangles of the metal insets formed by the square subdivided.
4. An envelope marked "The heights of the triangle."
5. Narrow cardboard strips (½ cm. wide, about 20 cm. long)
6. Red pencil.
7. White paper, glue, scissors, ruler.

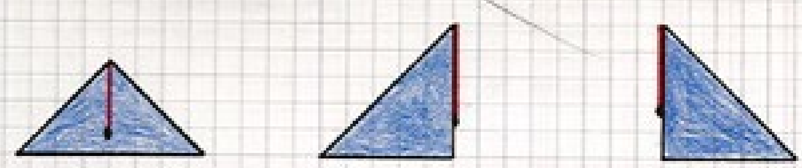
NOTE: The child has explored the altitude of the triangles that he constructed with the sticks. . .and only the easiest ones, that is those whose altitudes fall within the internal region of the triangle. The sticks were an impression. Now we reanalyze the heights with more precise material. So we use all the triangles in the Montessori material.

1. Begin with the blue constructive triangles. Remove the triangle that is used to form the trapezoid if it is a duplicate of one of the other. Show only one of the other 3 pairs.

1. We have 4 different triangles. Let's explore their altitude.

2. Begin with the right-angled isosceles triangle, showing first the internal altitude, then rotating the triangle in the stand to show the two altitudes which follow the sides.

2. This plumb line string represents the altitude of the triangle. Does it fall on the external or the internal part?



Now where does the altitude fall?
It falls neither on the external or the internal part; it follows the side.
And again: we see that the third altitude is neither in the external nor in the internal part.
This is a right-angled isosceles triangle.

3. ACTIVITY: the child draws the contour of this triangle in his geometry notebook.

4. Second triangle: the equilateral triangle. In the same manner, show that the equilateral triangle has three internal altitudes.
Then: the child draws the contour of this triangle.

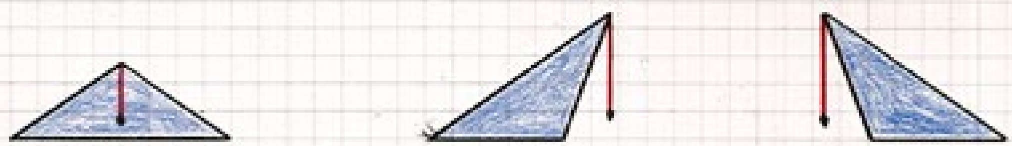
4. What triangle is this?
It is enough to call it an equilateral triangle. . .and we know the angles are equal.
We discover that the equilateral triangle has three internal altitudes.

5. Third triangle: right-angled scalene. Repeat the exploration of the three altitudes of this triangle on the stand. Then: the child draws the contour of this triangle in his notebook.

5. We discover that the right-angled scalene triangle has one internal altitude and two altitudes which fall neither in the external nor the internal part.

6. Fourth triangle: obtuse-angled isosceles. Repeat the discovery of the three altitudes, starting with the internal altitude position.

6. What is this triangle?
This triangle is more importamt according to its angles.
We discover that the obtuse-angled isosceles triangle has one internal altitude and two altitudes that fall on the external part.



7. Review the discoveries made about the altitudes of each of the triangles.

7. The right-angled isosceles triangle has one internal altitude and two altitudes that are neither internal nor external.
The equilateral triangle has three altitudes on the internal part.
The right-angled scalene triangle has one internal altitude and two altitudes that are neither on the internal nor the external part.
The obtuse-angled isosceles triangle has one internal and two external altitudes.

8. Take the drawer of triangles from the geometry cabinet. Make the two necessary substitutions to show the seven triangles. (acute-angled scalene in the top center row; and the obtuse-angled scalene to the side of the drawer).
EXAMINE FIRST ALL THE ACUTE-ANGLED TRIANGLES, showing the triangle in the wooden stand and using the plumb line to indicate the altitudes for each of the three bases.

8. Let's examine all the acute-angled triangles here.
First the isosceles: all the altitudes are internal.
Then the equilateral: all the altitudes are internal.
Then the scalene: all the altitudes are internal.

9. The child draws the contour of each of the acute-angled triangles he has examined and writes: In all the three acute-angled triangles (acute-angled isosceles, acute-angled equilateral, and acute-angled scalene), the altitudes are always lying on the internal region.

10. Examine the altitudes of the two right-angled triangles.

10. The right-angled scalene triangle has one internal altitude and one altitude which follows the long leg and one altitude which follows the short leg. Neither are on the external nor the internal part.
The same is true of the right-angled isosceles triangle.

THE ALTITUDE OF THE TRIANGLE. . .
Presentation. . .

11. The child draws the contour of the two triangles and writes: In both the right-angled triangles of reality, one scalene and one isosceles, the altitudes are arranged in this way: one is internal and two fall along the perpendicular legs.

12. Examine the altitudes of the two obtuse-angled triangles, the one scalene and the other isosceles.

12. We discover that the obtuse-angled scalene has one internal altitude and two external altitudes.
The same is true for the obtuse-angled isosceles triangle.

13. The child draws the contour of the two triangles and writes: In both obtuse-angled triangles (the scalene and the isosceles), I have discovered that the altitudes fall in the following way: one lies in the internal region and the other two on the external region.

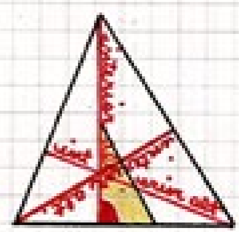
14. The Triangles of the metal insets are now considered, those formed by dividing and subdividing the square. Because they are thin, we must introduce one of the narrow cardboard strips into the slot in the wooden triangle holder to secure the inset in an upright position. The activities and the presentation here are the same as in our examination of the figures from the cabinet.

Presentation: Part II: Drawing the Altitudes

1. Note that we have learned in the exploration that the altitude is always perpendicular to the base. . .and so we can draw those altitudes.

1. Now we are going to consider the drawings we made during our analysis of the altitudes of the triangles.
We have already explored the angles with the measuring angle.
We said that the altitude is always a perpendicular line to the base.
What kind of angles do perpendicular lines form?

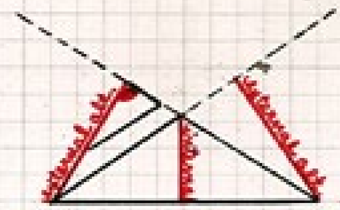
2. The child chooses one of his contour drawings with which to work. (It should not be the obtuse-angled triangle.) Then:
a) He shows that triangle (from the cabinet collection) on the stand.
b) He notes what he has written on his paper about the altitudes of this particular triangle.
c) He positions his figure drawn to show the same base as that of the figure he has shown on the stand.
d) He then shows the plumb line on the stand figure to indicate the altitude.
e) THEN, SLIDING THE MEASURING ANGLE ALONG THE BASE OF HIS DRAWN FIGURE, HE MOVES IT UNTIL HE COMES TO THE VERTEX.
f) He marks the base at that point and then, with a ruler, draws the altitude.
g) He then rotates both the figure on the stand and his paper and repeats the experience to draw the second and third altitudes.



3. He examines in this way the right-angled triangle, using one of his corresponding drawings. When he concludes each triangle, he writes the kind of altitudes on the figure.

THE ALTITUDE OF THE TRIANGLE. . .
Presentation #2: Drawing the Altitude

4. The child repeats the exercise with the obtuse-angled triangle. This presents the special problem of having to prolong two bases in order to draw the perpendiculars representing the altitudes.

4. 

In order to draw our second altitude, we can't follow the base because it is too short to guide the measuring angle to the vertex.
So I must prolong the base. The slot of the stand here is the prolongation of the base.
On the paper I draw the line of the slot. Now I can follow the base.
AND I MUST REPEAT THIS WITH THE THIRD ALTITUDE: Again the road ends, and with a ruler I must prolong the base.

When the child has finished this work with the obtuse-angled triangle, he adds to his statement about the obtuse-angled triangles: In order to draw the altitudes, it was necessary to prolong 2 bases.

Presentation #3: THE ORTHOCENTER: The Study of the Altitudes of the Triangle All Together.

1. Introduce the envelope of paper triangles marked "The Heights of the Triangle." It should contain many different sizes (small, medium, large) of each of the seven triangles.

1. In this envelope we can find each of the seven triangles, in many different sizes.
We are going to do the same paper work that we did with the contours you drew, but we are going to do it in a new way.

2. Take first the <u>acute-angled scalene triangle</u>. Hold the triangle in the wooden stand, show the plumb line from the vertex and MARK THE SPOT WHERE THE LINE FALLS WITH A RED PEN. Then the child uses a ruler to join that mark and the vertex to show the altitude.

2. When we show one of these triangles on the stand and locate the altitude with the plumb line, we will mark where that plumb line falls with a red dot.
Then we can join that mark with the vertex from where the line was dropped and we will have drawn the altitude.

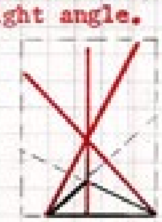
3. The child then rotates the paper triangle to the next base, repeats; and a third time. Point out the internal orthocenter.

3. I already know that all the altitudes of this triangle lie in the internal region.
The point where the altitudes meet is called the orthocenter.

4. The child repeats the experience with many acute-angled triangles and writes the conclusion: In all the acute-angled triangles, the orthocenter is internal.

5. Next the child uses those paper triangles which have right angles, beginning with a right-angled isosceles triangle. He repeats the experience as above. . .and then does this with many right-angled triangles.

5. We can write this conclusion:
In every right-angled triangle, the orthocenter coincides with the vertex of the right angle.



6. In the same exercise with the obtuse-angled triangles, the child meets the special problem of external altitudes which he cannot mark ON the triangle with the plumb line as a guide. So he pastes the triangle on a large piece of paper, then folds it up along the base, and on that paper he can mark the point through which he draws his altitude. He must refold the paper for the second base. The paper also provides for the external orthocenter.

We can't prolong the altitudes on this end because they are divergent.
We prolong them on the other side until they meet and conclude: In obtuse-angled triangles, the orthocenter is external.