MULTIPLES AND DIVISORS

Materials (for all exercises)

1. Any chain from the cabinet of powers with the set of arrows for skip counting.
2. The large box of bead bars.
3. The tables of multiples: A, B, and C.
4. Large peg board.
5. A wooden box with three divisions, containing many pegs in the three hierarchical colors.
6. A second identical box, but with pegs of three different colors. (This second group is not absolutely necessary.) Here the colors are white, black and yellow.
7. Slips of paper.
8. Tiny little bars.
9. Two large circles, one blue and one red: used for sets.

Presentation: The Multiples : THE CONCEPT

NOTE: Presented when the children have memorized all the multiplication tables and worked with ALL THE MATH MATERIALS PRESENTED TO NOW.

1. Take any chain from the cabinet of powers. Ask the child to match the "skip counting" arrows. Leave out the arrows that number the first bar.

2. Analyze those numbers shown on the arrows. Note that the base number is perfectly contained in each of them. HERE we use the short chain of 6.

2. 12---Do you think this number contains 6? YES---it contains 6 two times perfectly. 18---Is 6 also in 18? In 24? In 36? 6 is perfectly contained in each of these numbers.

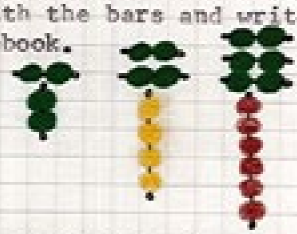
3. Give the concept of multiples. Give several other examples.

3. All of these numbers that contain 6 exactly are called the multiples of 6. All numbers which contain another number perfectly are called the multiples of that number. Can you give me a multiple of 5?
of 10?
of 8?

4. Review with an extended second and third period lesson.

4. Can you tell me which number 20 is a multiple of? of 5, of 2, of 10. Each of these three numbers is perfectly contained in 20.

5. The search for multiples with the bead bars: The child forms the multiples of 2 with the bars and writes them in his notebook.



5. Let's discover which are the multiples of 2 with the bead bars.
4 is a multiple of 2.
6 is a multiple of 2. . . .etc.

6. The child looks for the multiples of each quantity represented by one bead bar.

7. THEN he composes a number of one ten bar and another bar, searching for the multiples of the quantities from 11 to 19. In this exercise, the child no longer lays out the quantity horizontally; but can simple show vertically that: 22 is a multiple of 11, 33 is a multiple of 11, etc. by laying out 11 X 2 and 11 X 3 in bead bars vertically.

8. WHEN THE CHILD HAS UNDERSTOOD THE MULTIPLES WELL, use the printed form. Below the table there are lines where the child writes his own conclusions after each exercise with the form. Examples: a) He marks all the multiples of 3 (in a color) and writes: All the multiples of 3 are lying on the diagonal.
NOTE: With each exercise the child forms a different geometrical figure. Be sure to have many such forms, for the work is interesting and he can do this work with each quantity.

9. REVIEW: 12 is a multiple of which number? Tell me a multiple of the number 5.

T A V O L A   D E I   M U L T I P L I

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Le mie OSSERVAZIONI:

The multiples of 5 form vertical straight lines.
The multiples of 6 lie in ~~xxx~~ oblique straight lines.

Multiples and Divisors. . .

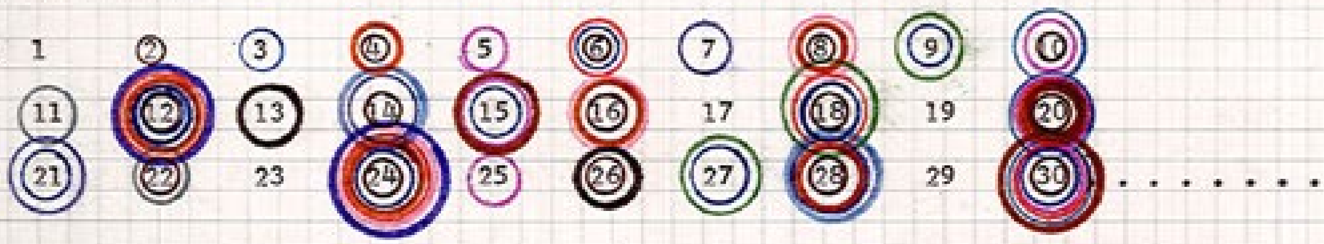
ADDITIONAL EXERCISES WITH THE MULTIPLES PRINTED FORM:

Exercise #1: The child marks a number and its multiples on the form in one color. Then he connects those with colored lines which may show straight lines or diagonals. Then he chooses another number; and, using another color, marks that number and its multiples, joining them also with colored lines. Finally, he notes those numbers where both lines intersect and circles those numbers. He is, in fact, showing those numbers which are multiples of both of the numbers he has chosen. He may do the same exercise with three numbers and three colors, which may necessitate using one colored circle for those points where two lines intersect and, perhaps, a bolder color where all three meet.

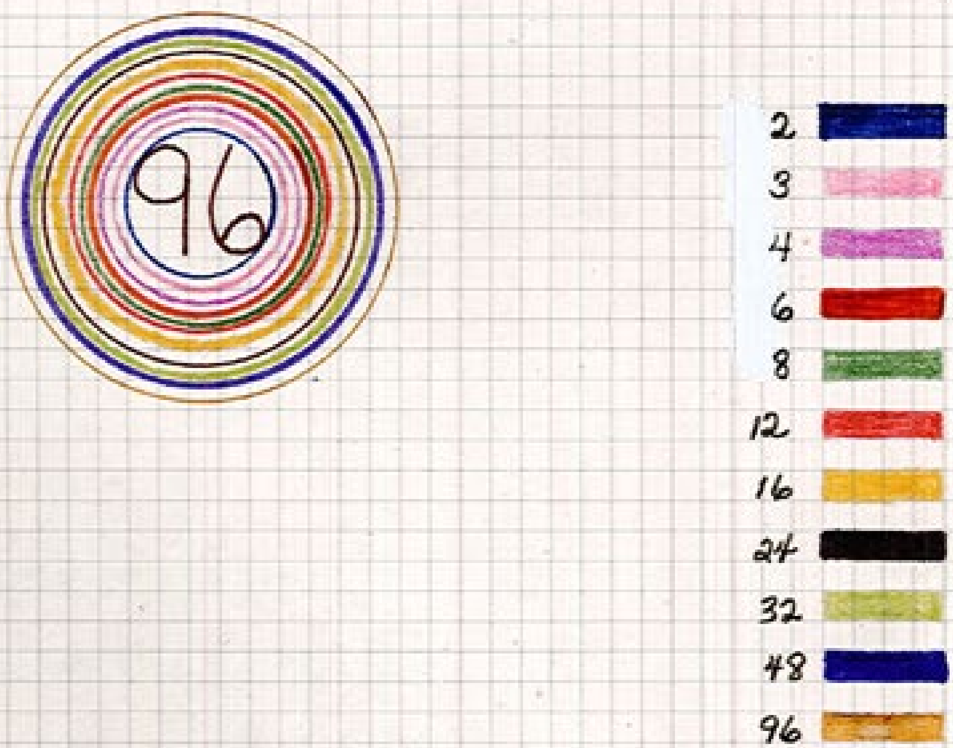
Exercise #2: Using the form, the child begins with 2 and its multiples, marking each in a particular colored circle. Then he chooses another color for 3, circling three in that color and all of its multiples. Then he shows 4 and its multiples in a third color. And proceeds in this way through all of the numbers. He discovers that some numbers are the multiples of many numbers.

Exercise #3: The amazing result of exercise #2 is that 96 is the multiple of ten numbers and thus has been circled in 10 different colors. The child may make a chart of this noteworthy multiple, or the teacher may present it. And then the chart of 96 is hung in the classroom.

#2:



#3:



2x1=	3x1=	4x1=	5x1=	6x1=	7x1=	8x1=	9x1=	10x1=
2x2=	3x2=	4x2=	5x2=	6x2=	7x2=	8x2=	9x2=	10x2=
2x3=	3x3=	4x3=	5x3=	6x3=	7x3=	8x3=	9x3=	10x3=
2x4=	3x4=	4x4=	5x4=	6x4=	7x4=	8x4=	9x4=	10x4=
2x5=	3x5=	4x5=	5x5=	6x5=	7x5=	8x5=	9x5=	10x5=
2x6=	3x6=	4x6=	5x6=	6x6=	7x6=	8x6=		
2x7=	3x7=	4x7=	5x7=	6x7=	7x7=			
2x8=	3x8=	4x8=	5x8=	6x8=				
2x9=	3x9=	4x9=	5x9=					
2x10=	3x10=	4x10=	5x10=					
2x11=	3x11=	4x11=						
2x12=	3x12=	4x12=						
2x13=	3x13=							
2x14=	3x14=							
2x15=	3x15=							
2x16=	3x16=							
2x17=								
2x18=								
2x19=								
2x20=								
2x21=								
2x22=								
2x23=								
2x24=								
2x25=								

Table A

Combining Tables A and B, we show all the multiples of the numbers 1 – 10 within the limits of 100. The child himself compiles the two tables. He may note with interest the repetition of certain numbers such as 96.

Table C

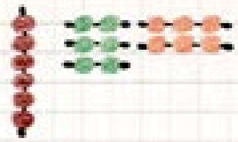
1	51 17x3 3x17
2	52 26x2 13x4 4x13 2x26
3	53
4 2x2	54 27x2 18x3 9x6 6x9 3x18 2x27
5	55 11x5 5x11
6 3x2 2x3 * So... 6 is a multiple of 3 and 2.	56 28x2 14x4 7x8 8x7 4x14 2x28
7	57 19x3 3x19
8 4x2 2x4	58 29x2 2x29
9 3x3	59
10 5x2 2x5	60 30x2 20x3 15x4 12x5 10x6 6x10 5x12 4x15 3x20 2x30
11	61
12 6x2 4x3 3x4 2x6	62 31x2 2x31
13	63 21x3 9x7 7x9 3x21
14 7x2 2x7	64 32x2 16x4 8x8 4x16 2x32
15 5x3 3x5	65 13x5 5x13
16 8x2 4x4 2x8	66 33x2 22x3 11x6 6x11 3x22 2x33
17	67
18 9x2 6x3 3x6 2x9	68 34x2 17x4 4x17 2x34
19	69 23x3 3x23
20 10x2 5x4 4x5 2x10	70 35x2 14x5 10x7 7x10 5x14 2x35
21 7x3 3x7	71
22 11x2 2x11	72 36x2 24x3 18x4 12x6 9x8 8x9 6x12 4x18 3x24 2x36
23	73
24 12x2 8x3 6x4 4x6 3x8 2x12	74 37x2 2x37
25 5x5	75 25x3 15x5 5x15 3x25
26 13x2 2x13	76 38x2 19x4 4x19 2x38
27 9x3 3x9	77 11x7 7x11
28 14x2 7x4 4x7 2x14	78 39x2 13x6 6x13 2x39
29	79
30 15x2 10x3 6x5 5x6 3x10 2x15	80 40x2 20x4 15x6 10x8 8x10 6x15 [illegible] 2x40
31	81 27x3 9x9 3x27
32 16x2 8x4 4x8 2x16	82 41x2 2x41
33 11x3 3x11	83
34 17x2 2x17	84 42x2 28x3 21x4 14x6 12x7 7x12 6x14 4x21 3x28 2x42
35 7x5 5x7	85 17x5 5x17
36 18x2 12x3 9x4 6x6 4x9 3x12 2x18	86 43x2 2x43
37	87 29x3 3x29
38 19x2 2x19	88 44x2 22x4 11x8 8x11 4x22 2x44
39 13x3 3x13	89
40 20x2 10x4 8x5 5x8 4x10 2x20	90 45x2 30x3 18x5 15x6 10x9 9x10 6x15 5x18 3x30 2x45
41	91 7x13
42 21x2 14x3 7x6 6x7 3x14 2x21	92 46x2 23x4 4x23 2x46
43	93 3x31
44 22x2 11x4 4x11 2x22	94 47x2 2x47
45 15x3 9x5 5x9 3x15	95 19x5 5x19
46 23x2 2x23	96 48x2 32x3 24x4 16x6 12x8 8x12 6x16 4x24 3x32 2x48
47	97
48 24x2 16x3 12x4 8x6 6x8 4x12 3x16 2x24	98 49x2 14x7 7x14 2x49
49 7x7	99 33x3 11x9 9x11 3x33
50 25x2 10x5 5x10 2x25	100 50x2 25x4 20x5 10x10 5x20 4x25 2x50

Again the child compiles Table C. In his first completion of this table, he finds all those combinations which give the numbers 1-100 excluding 1 and the number itself. Thus he discovers all those numbers of which the number is a multiple. *

Presentation #2: A Research of the Multiples of Two or More Numbers.

A. WITH THE BEAD BARS---discovering all the numbers of which one number is a multiple.

1. Select one quantity (6) and then show that number in as many combinations of beads as possible.



2. The child writes: 6 = 2 X 3
   6 = 3 X 2 OR
   6 is a multiple of 2 and 3.

3. In this way the child looks for all the numbers of which another number is a multiple:
   10 is a multiple of which numbers? OR Which numbers are contained perfectly in 10?

B. WITH THE CHARTS

1. Charts A and B: these actually form one chart, but have been shown as two for ease of handling the smaller size. On table A the highest product is 50; on table B the highest product is 100. The tables contain all the multiples of the numbers 2 - 10 to 100, showing each as a product. OBSERVE WITH THE CHILD: that the smaller the quantity, the more multiples it has. So 2 has many more than 10. These two tables are presented to the children as printed forms WITHOUT THE PRODUCTS. The work is done by 2 or 3 children. When they have completed both tables, they are then displayed in the classroom. On the form, the number is in red:

   10 X 1 = 10
   10 X 2 = 20
   10 X 3 = 30 etc. . . .

2. Chart C: On this chart we have the numbers 1 - 50 in one column, 50 - 100 in the second. This chart is presented as a form on which only these two columns of numbers is shown.

   a) The child examines all the numbers in the list, beginning at 1.
   a) How is 1 obtained? 1 X 1
   We cannot obtain it in any other way. So we don't write anything, but underline that number 1 in red.

   b) 2---
   b) 2 is obtained only by 2 X 1 and 1 X 2, so we underline it in red also.

   c) 3---
   c) The same for 3.

   d) 4--- He writes 2 X 2.
   d) Here we have numbers beside 1 and the number itself which give 4. Write 2 X2.

   e) In this manner the child fills in the whole page from 1 - 100.

   <u>1</u>
   <u>2</u>
   <u>3</u>
   4 2 X 2
   <u>5</u>
   6 2 X 3, 3 X 2 . . . .

   f) Consider the prime numbers when the child has completed the work. The state the main characteristic: the only multipliers of the prime number are 1 and the number itself.

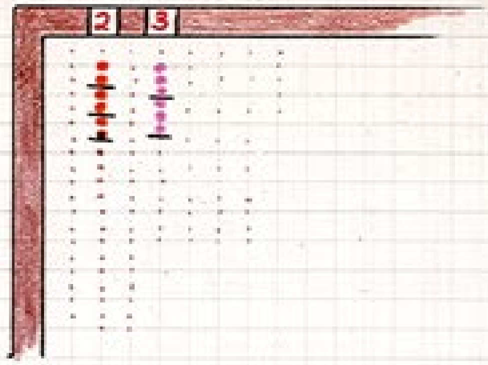
   g) The child checks his work with Charts A and B.

Pres #3: The Research of the Least Common Multiple: A Very Special Multiple

1. Using the peg board and pegs of colors (not the hierarchical ones), the search is for the multiples of 2 and 3 simultaneously. Groups of two pegs are laid out below a digit slip identifying the sets in the row. After each set is placed vertically, a small bar is laid to note the complete set. In a parallel column the set of the second number 3 is laid out. . .in the same fasion. So the column #1 increases by 2 each time and column #2 increases by 3. When the sets end parallely, we stop---6.

Multiples. . .

Presentation #3: The Research of the Least Common Multiple. . .



2. Analyze the work: How many times have the groups of 2 been repeated? 3
How many groups of 3? 2.
So 6 is a multiple of 2 and also of 3.

3. Give the concept of LEAST COMMON MULTIPLE:
The two is contained perfectly in the 6; the three is contained perfectly in the 6. SO. . .the number which contains more than one number perfectly has a special name. 6 is the SMALLEST number which contains both 2 and 3 perfectly.
It is the LEAST COMMON MULTIPLE OF 2 AND 3.

4. Introduce the abbreviation:

4. 6 is the smallest number that contains 2 & 3.
It is a multiple of both.
So it is common to the two.
It is the Least Common Multiple or the LCM.

5. Now add a slip for 4 on a third column: the search now is for the least common multiple of three numbers. We discover that it is 12. Then: 12 is the LCM of 2, 3, and 4.

6. Proceed to the LCM of numbers above 10, using the hierarchical colors:

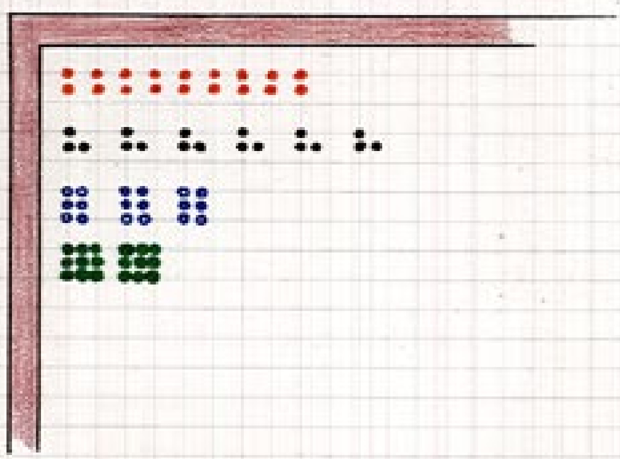
a) Place the little bars ONLY after the tens.
b) When we have shown 10 unit pegs of green, we must exchange for one ten, and that ten is included in whatever set is being formed. **
c) Again we continue the formation of sets until the columns reach a point of equality. . .then we stop and count the quantity in the columns.--the LCM.

SO. . .the LCM of the numbers 24 and 32 is 96.

Presentation #4: Research for the Divisor of a Number

NOTE: Here we start with the large quantity and we discover the smallest part contained perfectly; the opposite of the exploration in presentation #3.

NOTE: In this presentation, each peg represents one unit, no matter what colors are used.



1. Ask the child to select a number: 18. Then count out 18 pegs of a color and begin on the board by forming groups of 2 to discover if 2 is perfectly contained in 18. It is a divisor of 18.

2. With another color, take 18 and form groups of 3. It is a divisor of 18.

3. Proceed: groups of 4 in a third color cannot be formed because we have two remaining. SO it is not a divisor of 18. Remove the groups from the board.

4. Try 5---it doesn't work. 6 does---We form 3 groups of 6 with the 18 pegs so 6 is perfectly contained in 18 and it is a divisor of 18. 7 doesn't work. Nor 8. But with 9 we find we can form two groups and we have discovered another divisor. 10 is too large because with two groups we would have more than 18.

5. CONCLUDE; The divisors of 18 are 2, 3, 6, and 9.

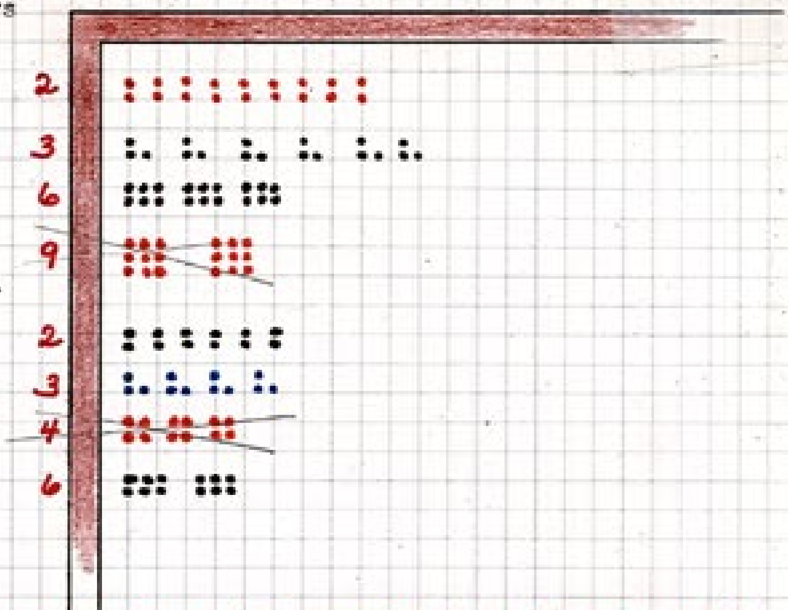
NOTE: We know we could form one group of 18, but we don't need to show that.

Presentation #5: The ~~Largest~~ Greatest Common Divisor: When the children have worked well with presentation #4, and understand the process of discovering the divisors of one number. .

1. Now we search for the common divisors of two numbers.
   a) First the divisors are explored for each of the numbers: here 18 and 12.
   b) When the work is completed, we observe which divisors are common to both numbers?
   c) Remove those pegs which show the divisors which are not common--- here 9 for 18 and 4 for 12.
   d) Now we show those divisors which are common to both numbers: 2, 3, and 6.
   e) Then we observe which is the greatest common divisor. . .

   6

   f) 6 is the ~~largest~~ greatest common divisor of the numbers 18 and 12.



2. Note now the important difference between the Least Common Multiple and the ~~Largest~~ Greatest Common Divisor.

   Examples: 6 is the GCD of the numbers 12 and 18.
   24 is the LCM of the numbers 2, 4, and 6.

   The Least Common Multiple is the SMALLEST number which contains perfectly 2 or more numbers.
   The ~~Largest~~ Greatest Common Divisor is the largest number which divides 2 or more numbers.
   Which one is usually the largest number? The LEAST Common Multiple.

3. Note the differences in the two abbreviations, writing first the two phrases and then noting the formation of the abbreviations: LCM and LCD.

Presentation #6: A Passage to Abstraction

At a certain point the child abandons the material. This is the exercise that brings him to that point.

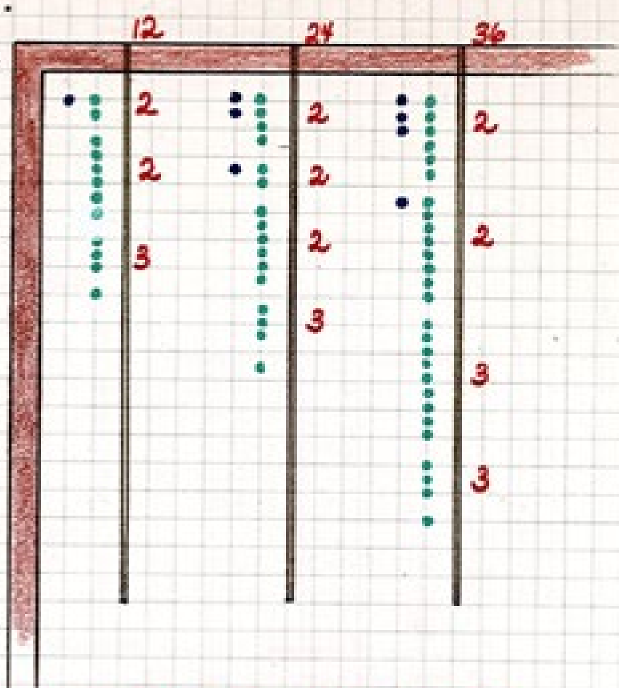
Now the child again fills in Chart C. He begins by underlining the prime numbers in red and showing the product operations for 4 and 6.

THEN. . .when he gets to 8, we explain that now we are going to form the products only with the prime numbers. So instead of 4 X 2 and 2 X 4 for 8, we write: 2 X 2 X 2. We are showing the 4 as 2 X 2 because it is not a prime number.

The child continues to complete chart C in this way, using the first Chart C which he filled out as a reference, a guide to those operations which he showed and which now must be shown with the prime numbers. He discovers that now each number is shown as only ONE product. On his first Chart C, 32 = 16 X 2, 2 X 16, 4 X 8, 8 X 4.
Now he writes32 = 2 X 2 X 2 X 2 X 2

1		11	
2		12	2 X 2 X 3
3		13	
4	2 X 2	14	2 X 7
5		15	3 X 5
6	2 X 3	16	2 X 2 X 2 X 2
7		17	
8	2 X 2 X 2	18	2 X 3 X 3
9	3 X 3	19	
10	2 X 5	20	2 X 2 X 5

CONCLUSION: We have shown now the prime factors of each number from 1 - 100.



2. Now we consider the prime factors of three numbers, here 12, 24, and 36.
a) We form each quantity with the hierarichically colored pegs, showing the number to the right of the peg display, above the board.
b) We are looking for prime factors, thus we begin by dividing by the prime numbers excluding 1 which does not help.
c) So we divide by 2 as long as it is possible, with each division noting the divisor with a slip of paper on the board and forming the product with pegs.
d) We ask: Is 12 divisible by 2?
YES---I obtain 6. . . .
e) When the product is 1, the division must stop. We can now read our prime factors as that group of numbers by which we divided the number.
f) NOTE: the black lines shown are cardboard strips.

3. CONCLUSION: WE HAVE found all the prime factors of the numbers 12, 24 and 26.

4. The child writes: 12 = 2 X 2 X 3
24 = 2 X 2 X 2 X 3
36 = 2 X 2 X 3 X 3

5. Analyze with the child the difference in the repetition of the factors in each of the numbers.

5. These numbers have common prime factors, but the prime factors are not repeated the same number of times in each.
How many times is 2 repeated in 12? 24?

6. Now we look for the Least Common Multiple and the ~~Largest~~ Greatest Common Divisor of the three.

6. First we find the number in which 2 is repeated the greatest number of times.
It is 24 where it is repeated three times.
In which number is 3 repeated the greatest number of times? 36---repeated twice.
Now by showing the product of the factors (prime) the most number of times in any one of the numbers, we discover the LCM

LCM = 2 X 2 X 2 X 3 X 3 = 72
How do we know 72 contains each number perfectly?
We divide 72 by each one.
72 ÷ 12 = 6
72 ÷ 24 = 3
72 ÷ 36 = 2

Now we want to discover the GCD.
Will it be larger or smaller than the GCM?
In order to find the GCD we take the prime factors where they are repeated the least number of times and are common to all numbers.

2 is repeated the least number of times in 12---twice---and that prime factor is common twice to each of the numbers.
3 is repeated the least number of times in 12---once---common to each number. SO
the GCD = 2 X 2 X 3 = 12

Presentation #7: The Calculation With the Powers: To Abstraction

Now we ask: How do you write 2 X 2 in a different way? Instead of saying 12 = 2 X 2 X 3 we can say 12 = 2^2 X 3. AND 24, written with the powers is 24 = 2^3 X 3. The child writes:

12	2	24	2	36	2
6	2	12	2	18	2
3	3	6	2	9	3
1		3	3	3	3
		1		1	

$12 = 2^2 \times 3$
$24 = 2^3 \times 3$
$36 = 2^2 \times 3^2$

Then, finding the prime factors shown the greatest number of times in the numbers we find:
LCM = $2^3 \times 3$

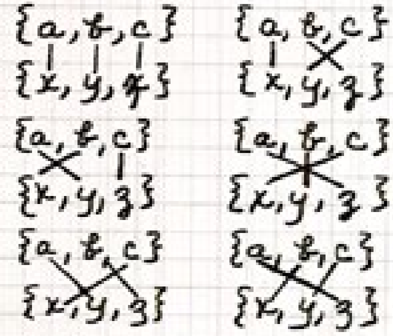
and taking the prime factors the least number of times in any number:
GCD = $2^2 \times 3$

FINDING THE LCM & GCD with the SETS

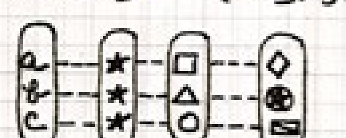
If the children have had exercises with the theory of sets, we begin our investigation to find the GCD with the sets; that is, the set work would be given prior to the previous presentations of the LCM and GCD. If they have not had work in the theory of sets, we now present these set exercises.

NOTE: NEW MATH SET TERMINOLOGY

1. A **set** is a collection of things considered as a single entity.
2. The things contained in a given set are called **members** or **elements of the set.**
3. The **braces** denote the set. { }
   The names of the members of the set are listed, separated by commas, and the enclosed in braces. Capital letters are used to denote sets.
   W = {Monday, Tuesday, Wednesday, Thursday, Friday, Sat, Sunday}
4. We abbreviate the phrase "is a member of" by using the Greek letter **epsilon.** ϵ A slash line negates the meaning of that symbol which becomes "is not a member of." $\notin$
   Monday $\in$ W
   May $\notin$ W
5. The **empty set** is the set that contains no members. It is denoted by the sign $\emptyset$, a letter from the Scandinavian alphabet.
   We also indicate the empty set by empty braces.
   A = the months containing more than 50 days.
   A = $\emptyset$ or A = { }
6. "Set A is a **subset** of set B" means that every member of set A is also a member of set B.
7. "Set A is a **subset** of set B if set A contains no member that is not also in set B.
   We abbreviate the phrase "is a subset of" with the symbol $\subset$.
   $A \subset B$ if for every $x \in A$ then $x \in B$.
8. **The empty set is a subset of every set.**
9. **Every set is a subset of itself.**
   R = {a, b, c, d, e}
   S = {a, c, e}
   $S \subset R$ and $R \not\subset S$
10. If A and B are names for sets, **A = B** means that set A has identically the same members as set B, or that A and B are two names for the same set.
    A = {r, s, t, u}
    B = {s, u, r, t}
    A = B
11. When K and M do not contain identically the same members, we say K is not equal to M or K $\neq$ M.
12. **There is a one-to-one correspondence between sets A and B if every member of A is paired with one member of B and every member of B is paired with one member of A.** The existence of a one-to-one correspondence has nothing to do with the way in which the pairing is done.



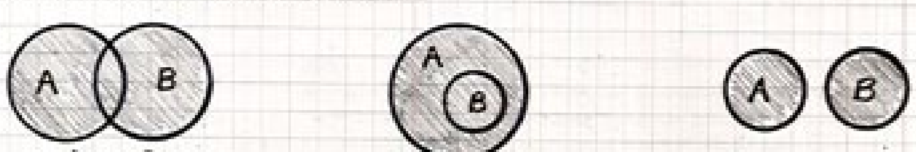
13. Two sets are **equivalent** if there is a one-to-one correspondence between the two sets. The two sets may have different members, but the one-to-one correspondence exists.
14. Equivalent sets: **The thing that is alike about these sets is called the number three.** Other sets belong to this collection: the set of wheels on a tricycle, the set of people in a trio, the sides of a triangle.



15. The number has many names---III, 2 + 1, 3, and many more. Each of these names is called a **numeral. A numeral is a name for a number.**
16. With every collection of equivalent sets is associated a number, and with each number is associated a simplest numeral.

Set	Number	Simplest numeral
{ }	zero	0
{a}	one	1
{a,b}	two	2
{a,b,c}	three	3
⋮	⋮	⋮

17. **The union of set A and set B**, denoted by A ∪ B, is the set of all objects that are members of set A, of set B, or of both set A and set B. **VENN DIAGRAMS:** a closed figure used to denote the set of all points within the figure. Here the shaded region in each illustration indicates A ∪ B. The union of two sets includes the members in both the sets.



A ∪ B A ∪ B A ∪ B

Finding the LCM & GCD with the SETS. . .
New Math terminology. . .

18. The intersection of set A and set B, denoted by A ∩ B, is the set of all objects that are members of both set A <u>and</u> set B. The shaded region in each of the following illustrations represents A B.

19. Set A and set B are called disjoint sets if they have no members in common. OR set A and set B are disjoint sets if A ∩ B = ∅.

Presentation: LCM & GCD with the Sets

Material: A large red circumference and a blue one.

1. Ask the child to find the LCM & GCD of two numbers in his usual manner of calculation:

8	2
4	2
2	2
1	

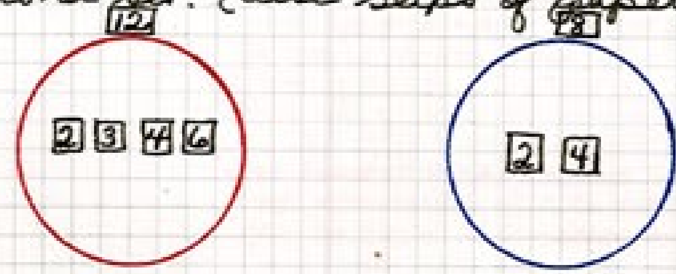
$8 = 2^3$

12	2
6	2
3	3
1	

$12 = 2^2 \times 3$

$LCM = 2^3 \times 3$
$GCD = 2^2$

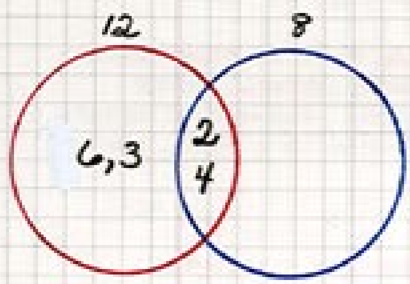
2. Using the two circumferences, show all the divisors of one number in first set and all divisors of second in second set. (Use slips of paper)



A = { the divisors of 12}
A = {2, 3, 4, 6}

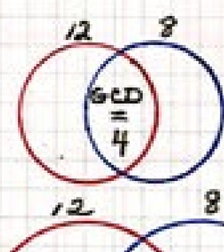
B = {the divisors of 8}
B = {2, 4}

3. Show the common divisors as the intersection of A and B.

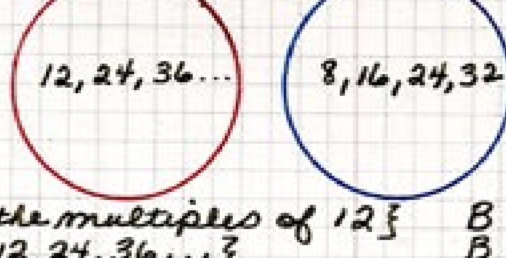


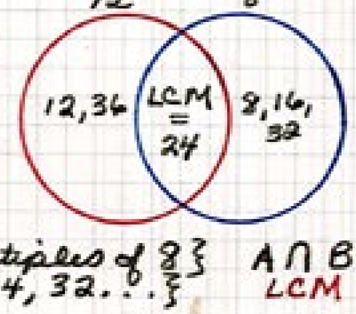
A ∩ B = {2, 4}

4. Remove the other numbers, showing the intersection GCD = 4 or 2^2.



5. LCM:





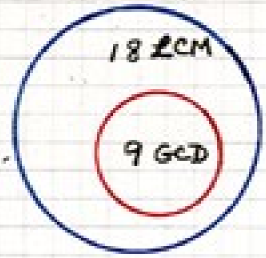
A = {the multiples of 12}
A = {12, 24, 36...}

B = {the multiples of 8}
B = {8, 16, 24, 32...}

A ∩ B = 24
LCM = 24

Finding the LCM and GCD with the Sets

6. When we have two numbers, one of which is the multiple of the other, we can show the smaller number as a subset of the larger. The largest will then be the LCM and the smallest the GCD.



$A \subset B$

EXERCISES: It is important that we provide VERBAL PROBLEMS which illustrate the usefulness of the LCM and the GCD.

Three examples:

1) A rectangular piece of paper measures 18 X 24 cm. This paper can be subdivided into what number of equal squares in such a way that we obtain squares of the greatest possible measure?
We need the GCD because we know that we must have a measure of each side equal to the others. We are searching for that divisor common to both. And we want the greatest possible measure.
NOTE: Having used those prime factors common to both in the greatest number to form the GCD, the product of the remaining factors of the two numbers gives the number of squares.**
** $3 \times 2 \times 2 = 12$

18	2
9	3
3	3**
1	

24	2**
12	2**
6	2
3	3
1	

GCD $= 2 \times 3 = 6$

$24 = 6 \times 4$

$18 = 6 \times 3$

6^2	6^2	6^2	6^2
6^2	6^2	6^2	6^2
6^2	6^2	6^2	6^2

2) We have paper rectangles measuring 6 X 10 cm. We want to arrange them so that they cover a square. How many do we need to cover a square? What size will the square be?
We need the LCM. By multiplying the factors, we obtain the side of the square.

10	2
5	5
1	

6	2
3	3
1	

LCM $= 5 \times 2 \times 3$ or 30 cm.

Then $30\text{cm} \div 10 = 3$
$30\text{cm} \div 6 = 5$

$10 \times 3 = \boxed{30 \text{ cm}}$

$6 \times 5 = \boxed{30 \text{ cm}}$

Note: We can also take the GCD, 2, and divide the product of the two numbers to get the side.
$\frac{60}{2} = 30$

The remaining factors gives the number of squares:
$3 \times 5 = 15$

DIVISIBILITY (following Multiples)

There are divisions with a remainder and those without a remainder. Now we are concerned only with those divisions without a remainder. We want to know if one number is divisible by another WITHOUT carrying out the division. The means which will enable us to do this are called the criteria of divisibility.

Presentation: Divisibility by 2

Material: The materials of the decimal system.

1. Write: 1126. And show the quantity of decimal materials which represent that number: one cube, one square, two ten-bars, and six unit beads.

2. Divide the quantity into two equal groups. Exchange first the cube, and proceed until the entire quantity is divided.

   2. Let's see if we can form with this quantity two equal groups. Is is possible? YES

3. The child writes the number in his notebook, underlines the last digit, and writes YES.

4. The child then makes a list, beginning with the possibility of ADDING ONLY ONE UNIT TO THE QUANTITY, THEN ANOTHER, stating whether or not equal groups can be formed.

5. Use a variety of other examples, taking the corresponding decimal materials for the proof. The child continues his list.

1,126	yes
1,127	no
1,128	yes
70	yes
15	no
16	yes

6. Analyze with the child those quantities where he has written yes.

   6. When we wrote "yes" beside the quantity, what was the last digit? Even or Odd?

   CONCLUDE: We can conclude that all those quantities whose last digit is even or zero are divisible by 2.

NOTE: Preparations for Even and Odd numbers: The Counters, Addition Memorization Chart #5, Decanomial construction with the squares: Game of Substitution.

Presentation: Divisibility by 4

1. Begin with the number 816: form the quantity with the decimal system quantities. Then, beginning with the largest quantity, the eight hundred squares, divide the quantity into four equal groups.
   NOTE: It is evident in this work how complete the children's understanding must be of the hierarchical changes with the decimal system materials.

   Let's see if we can form four even groups with the quantity 816. We CAN divide the hundreds into four even groups. But it is not possible with the one ten-bar. SO WE MUST EXCHANGE THE TEN-BAR FOR TEN UNITS---then we can make four even groups with the units. Then we have formed four equal groups with 816.

2. The child writes the number in his notebook and underlines the LAST TWO DIGITS---and YES he will see why at a later point.

3. He adds another unit bead and writes the number 817---he knows it is not possible now to have four equal groups. He adds one unit more two more times---then finally comes to the fourth additional unit which gives him 820---he can make four equal groups and he underlines those last two digits. The child now sees that a new rule must be discovered for the divisibility of 4.

816	Yes
817	No
818	No
819	No
820	Yes

At this point, we can conclude that the rule for divisibility is not that same one as for the 2. Both 16 and 18 are even, but 18 is NO: there must be something else.

4. He continues the exploration with other numbers, dividing the quantities into four equal groups if possible. And writes:

128	Yes
130	No
100	Yes

5. The children continue to write numbers and form the quantities, dividing those quantities into four equal groups in order to discover the secret. Eventually he sees that the secret is in the LAST TWO DIGITS. He can now form the rule:
   Those quantities whose last two digits are divisible by 4, are divisible by 4. That is, if the last two digits are divisible by 4, the whole quantity is divisible by 4. AND if the last two digits are ZERO, the quantity is divisible by 4.

Presentation #3: Divisibility by 5

In the same way, the child begins with a number: 125. And forms that quantity with the decimal system materials. Then he divides the quantity into equal groups (5). Here he underlines ONLY THE LAST DIGIT OF THOSE QUANTITIES WHICH ARE DIVISIBLE BY 5; that is, those quantities which he is able to divide into five equal groups. He compiles a list:

125	Yes
126	No
127	No
128	No
129	No
130	Yes
15	Yes
475	Yes

And he forms the rule: A number is divisible by 5 only if the last digit is a 5 or a zero.

Presentation #4: Divisibility by 25

1. Begin by forming 25 with two ten-bars and five units beads. Write a numeral card and show it above the quantity: 25.

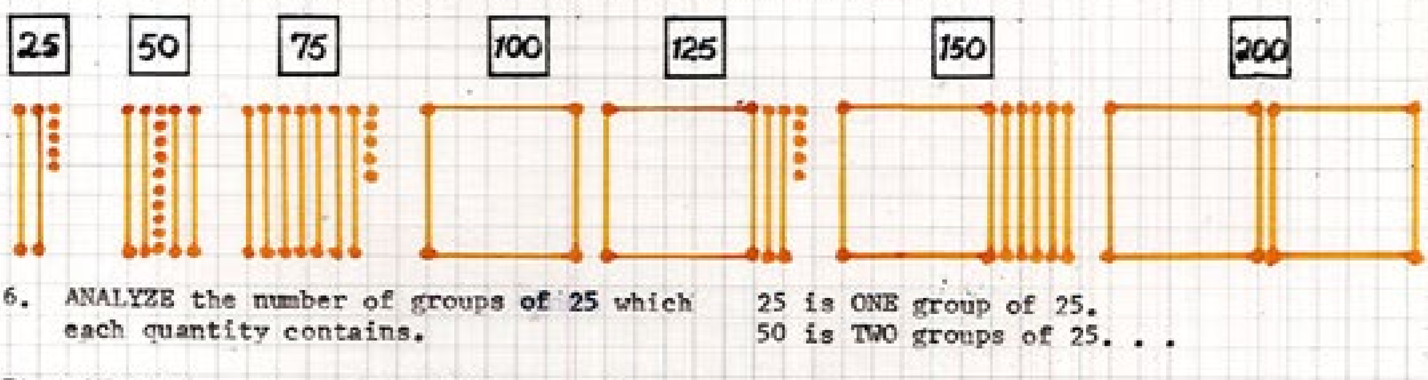
2. Then form two groups of 25 with the decimal system materials. Put them together. And identify the quantity that results: 50.

3. Then form three groups of 25: Again identify the quantity with a card.

3. We know that two groups of 25 gives 50. So we can use five 10-bars and add one more group of 25. The resulting quantity is 75.

4. Show four groups of 25 in the same way: now, because the child knows that two groups of 25 make 50, he knows that four groups will give 50 X 2 or 100: he uses a square.

5. Continue: showing five, six, seven and eight groups of 25:



6. ANALYZE the number of groups of 25 which each quantity contains.

25 is ONE group of 25.
50 is TWO groups of 25. . .

7. NOTE that we have stopped because the pattern will continue to repeat.

8. FORM THE RULE: A quantity is divisible by 25 if the last two digits are: 25, 50, 75 or TWO ZEROS.

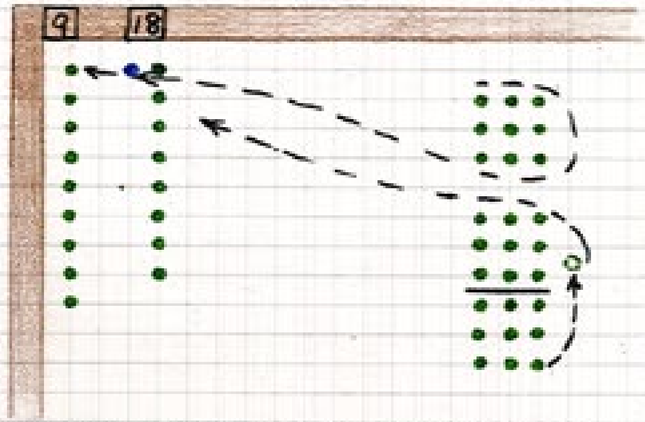
Divisibility by 9. . .

Presentation #5: Divisibility by 9: Transfer to Hierarchical Representation of the Quantities

Material
1. The peg board.
2. The pegs in hierarchical colors.
3. Slips of paper.
4. The charts of multiples, particularly Chart C.
5. The Multiplication Chart I.

NOTE: Even though this seems a step backwards in terms of the divisibilities considered, it is an important step forward. There is great power in the number 9, so we begin with a psychological preparation:

1. The Preparation: "Nine is the square of another important number: 3. It is also the last number of the natural series of numbers. This time we will consider 9 as the square of 3.
Have you done some work in which three is an important number, in which 3 has an important role? When you formed the polygons, three was an important figure. The triangle with three sides was an important figure, because with it you were able to form other figures. (Constructive triangles) So now we will look at 9 as a MULTIPLE OF 3.



2. On the peg board, construct the square of 3. "The square of 3 is formed of 9 units."

3. Then FORM A VERTICAL COLUMN OF THE 9 PEGS, placing a numberal card 9 at the top.

4. Next form TWO square of 3: two groups of 9. Take one peg from the lower square to join with the 9 pegs of the top square, thus forming 10---EXCHANGE FOR ONE BLUE PEG. Then show the blue peg and the remaining eight green ones in a vertical configuration next to the 9 on the board.

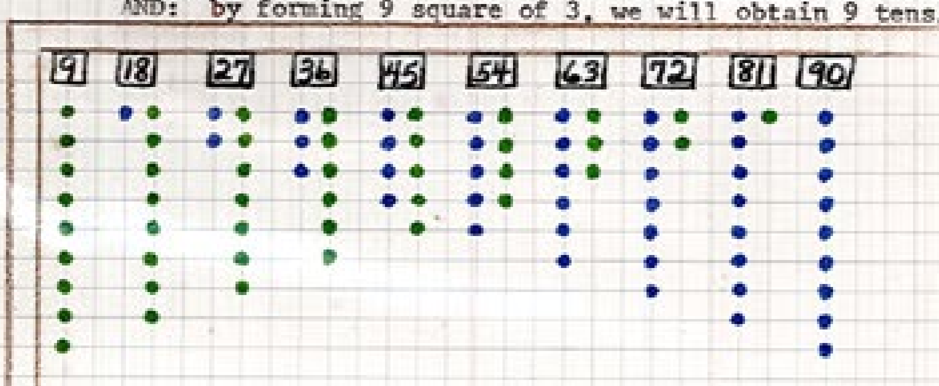
NOTE: The formation of square is on the far right of the board; the vertical quantities shown on the left.
AND in the transformation to the hierarchical pegs, we always take the single units from the bottom square, thus showing the decreasing number of units remaining.

5. Continue the work, forming three squares, and transfering once the exchange for TWO tens has been made. Then four squares, . . five square---to nine squares.

6. When the work has been completed through FOUR SQUARES, STOP AND ANALYZE THE FORMATIONS. Then continue the following quantities in order to verify the deduction.

So far we have formed four square of 3.
With one square, we had 9 units.
With two squares, we had ONE ten and 8 units.
As soon as we had one 10, our column of units decreased by 1. With two 10s, our column of units decreased by 2. With three 10s, it decreased by 3.

WHAT WILL 5 GROUPS OF 3 GIVE: ONE MORE TEN AND ONE LESS UNIT.
AND: by forming 9 square of 3, we will obtain 9 tens.



7. ANALYZE the work done. NOTE that at 45 the digits reverse and we have the identical pattern of digits from that center point in either direction.

9 18 27 36 45 54 63 72 81 90

From 9 to 45, the tens increase by one and the units decrease by 1; from 45 on that pattern continues, but the inverse pattern of digits can be seen.
AND: We see that the sum of the digits of each number is 9.

8. FORM THE RULE: All numbers whose digits give the sum of 9 are divisible by 9. AND since we formed all of these numbers by using the squares of 3, we also know that each of those numbers divisible by 9 is also divisible by 3.

9. Using the MULTIPLES CHARTS, we make reference to the multiples of 3 and 9. On those forms where the child circled multiples, he can observe that all those multiples which he circled for 9 were also multiples of 3. With the Multiples CHARTS A, B and C he can further verify this.

10. COMPARING THE WORK SHOWN ON THE PEGBOARD WITH THE MULTIPLICATION CHART I (The Pythagorean Tables) the child can also refer to the table of 9, seeing that 18 is formed of 9 X 2 OR two squares of 3. . .We can also see this comparison in the printed forms for multiplication which the child used in his memorization work.

Presentation #6: 9 As the Result of (10 - 1)
Now we consider the divisibility of 9 again, but as the result of (10 - 1).
The aim is the same: to discover if a number is divisible by 9.

Material: The decimal system materials.

1. Present the 1000-cube. In order to divide it into 9 equal groups, it must be exchanged for squares.	1. Is 1,000 divisible by 9? We must form 9 equal groups. NO. We have ONE hundred square remaining.

EXCHANGE.

2. Distribute then the ten ten-bars , and ONE ten-bar remains. EXCHANGE. When the ten unit beads have been distributed among the nine groups, there is ONE remaining.

3. CONCLUSION: 1,000 is not divisible by 9, BUT if we take away 1, the quantity is divisible by 9. THEN:

$$1{,}000 - 1 = 999$$
$$100 - 1 = 99$$
$$10 - 1 = 9$$

THE ADDITION OF THE DIGITS IS ALWAYS DIVISIBLE BY 9. The three numbers resulting when we take one away from 1,000, 100, and 10 are all divisible by 9.

****

4. Write the number 4,653. Show it decomposed. The, still considering 9 as (10 - 1), we see that in order to render 50 divisible by 9, we must subtract 5. Then we have 45 with a remainder of 5. THREE, FROM THE BEGINNING, IS CONSIDERED A REMAINDER. In the same way, we can make the 600 and the 4,000 divisible by 9.

$$\begin{array}{rl} & 3 \\ 50 & = 10 \times 5;\ 9 = (10-1);\ 50 - 5 = 45 \\ 600 & = 100 \times 6;\ 99 = (100-1);\ 600 - 6 = 594 \\ 4{,}000 & = 1{,}000 \times 4;\ 999 = (1{,}000-1);\ 4000 - 4 = 3996 \end{array} \qquad \begin{array}{r} 3 \\ 5 \\ 6 \\ 4 \\ \hline 18 \end{array}$$

5. ADD THE REMAINDERS: Since 18 is a number divisible by 9, we know that the whole quantity is divisible by 9.

6. But we can look for those remainders in another place: We see that they are those digits seen inthe decomposition of the number which are the actual digits of the number. Therefore, IT IS ENOUGH TO ADD THE DIGITS OF THE NUMBER TO DETERMINE ITS DIVISIBILITY BY 9. If the sum of the digits of the number is divisible by 9, then the whole quantity is divisible by 9.

Divisibility by 9

Presentation #7: Consideration of 9 in (10 = 9+1)

The question is still: Is 4653 divisible by 9?

1. Decompose the number: 4,000 is equal to 4 x 1,000. If I consider 1,000 as (999+1), then in order to make 4,000 divisible by 9, I will multiply 4x999... which gives 3,996. With a remainder of 4

3

	Remainders
50 = 5 x 10; 10 = 9+1; 5 x 9 = 45	3
600 = 6 x 100; 100 = 99+1; 6x99 = 594	5
4,000 = 4 x 1,000; 1,000 = 999+1; 4 x 999 = 3,996	6
	4
	18

2. Add the remainders. We see that the remainder 18 is divisible by 9. So, having rendered the other portion of the quantity divisible by 9, we can now say that the whole quantity 4,653 <u>is</u> divisible by 9.

3. Note the identity of the remainders and the digits of the number itself: so we can return to our original solution for divisibility by 9: the addition of the digits.

Proof of 9: A Check for Multiplication (not absolutely reliable)

1. We begin with a multiplication: we want to check the product. 372 x 455 = 169,260

2. Add the digits of the three terms: we see that NONE of the three is divisible by 9.

2. We know that when the sum of the digits of a number is divisible by 9, the whole quantity is divisible by 9.

3. Isolate those digits which give 9. 372 x 455 = 169,260

4. Analyze the remaining digits.

4. What are the remaining digits called?

3 is the remainder of 372 ÷ 9 = 41 r. 3
5 is the remainder of 455 ÷ 9 = 50 r. 5
We can show the same result with 6 in the product...

5. A. Multiply the remainders of the multiplicand & multiplier as shown in the top quadrants.
B. Add the digits of that product.
C. Their sum should give the remainder of the product if the multiplication is correct.

Remainder of multiplicand	x	Remainder of multiplier	
3	x	5	= 15
and 15 → (1+5)		If product is correct the remainder* will be the same number.	
6	→	6	
		*remainder of product.	

Note: 15 → 1+5 = 6 is actually 15 ÷ 9 = 1 r. 6. That is, the numbers here considered (3, 5, 6) are those remainders of the divisions of the whole quantities by 9... and it is important to show this.

6. Continue with other examples.

7. Formulate the rule: To check a multiplication, we first find the remainders of the multiplicand and multiplier divided by 9. The product of these remainders and the product of the multiplication, when divided by 9, must give equal remainders.

Note: Children learn this proof early; use in bank game & checkerboard.

Presentation: Divisibility by 11

1. Begin by taking a ten-bar of the decimal material. It is not divisible by 11, but then we ADD ONE MORE UNIT BEAD, and EXCHANGE THE TEN-BAR FOR UNITS. Now we can make 11 groups equally. Write as power: $10^1 + 1 = 11$

1. Neither 10 nor the powers of 10 are divisible by 11. But they can be divisible if we add 1. If we consider 10 as a power of 10, we have 10^1. When we add one unit, we have a quantity divisible by 11: $10^1 + 1 = 11$

2. Ask the child to notice the exponent: TAKE the hundred square. To divide it into 11 equal groups, we must exchange for 10-bars, then one ten-bar for units and WHEN WE ADD ONE UNIT TO EACH OF THE NINE TEN-BARS, we have nine groups of 11----with one remaining. That is, we must TAKE AWAY ONE TO RENDER THE 100 DIVISIBLE BY 11.

2. How can 100 be written? 10^2 We discover that we can form 9 groups of 11. That is, if we have 100 and we TAKE AWAY ONE UNIT, we have a quantity divisible by 11.

$$10^2 - 1 = 99$$

3. Write 1,000 as a power of 10. Take the thousand cube, showing that ONE UNIT MUST BE ADDED TO FORM 91 groups of 11.

3. $1{,}000 = 10^3$ $\quad 10^3 + 1 = 1{,}001$

4. Write 10,000 as 10^4. By a simple division the child discovers that the quantity is divisible by 11 if he TAKES AWAY ONE.

$$10^4 - 1 = 9{,}999$$

5. EXAMINE THE EXPONENTS: Formulate the rule:

$$\left.\begin{array}{l} 10^1 + 1 = 11 \\ 10^3 + 1 = 1{,}001 \\ 10^5 + 1 = 100{,}001 \end{array}\right\} \div 11$$

$$\left.\begin{array}{l} 10^2 - 1 = 99 \\ 10^4 - 1 = 9{,}999 \\ 10^6 - 1 = 999{,}999 \end{array}\right\} \div 11$$

When the exponent of 10 is odd, the quantity will be divisible by 11 if we add a unit. If the exponent is even, the quantity will be divisible by 11 if we take away one unit. That is, all the powers of 10 with odd exponents are divisible by 11 if we add one; and those with even exponents if we take away one.

6. SO. . .IS 23,485 divisible by 11?

	+	−
5		−5
$80 = 10^1 \times 8$	+8	
$400 = 10^2 \times 4$		−4
$3{,}000 = 10^3 \times 3$	+3	
$20{,}000 = 10^4 \times 2$		−2
	+ 11	− 11

A. In order to make 80 divisible by 11, we must add 8, indicated by the even exponent. . . . and so on . . .

B. Add the + and − columns. Now we subtract the positive remainder quantity (+ 11) from the negative remainder quantity. Drop the + and − signs. So 11 − 11 = 0. The whole quantity is divisible by 11.

C. Note that these remainders in both columns give the digits of the number, so we can mark the digits of the quantity itself with + and − signs. . . indicating the positive and negative remainders STARTING FROM THE RIGHT with a − sign and alternating.

$$\overset{-}{2}\,\overset{+}{3},\overset{-}{4}\,\overset{+}{8}\,\overset{-}{5}$$

$$11 - 11 = 0$$

7. Give the rule: If we add the digits marked with plus signs and minus signs in this way, the difference of the sums (taking the negative quantity sum ALWAYS AS THE MINUEND) will be 11, zero (0), or a multiple of 11 if the number is divisible by 11.

8. Special case: $\overset{+}{4}\,\overset{-}{3}\,\overset{+}{8}\,\overset{-}{5}$. The minuend (5+3) is less than the subtrahend in our proposed subtraction proof. So we add 11 to the minuend:

$[(5+3)+11] - (8+4) = 19 - 12 = 7$ NO. Not divisible by 11.

Divisibility...
by 11...

9. Conclusion: We see that divisibility by 11 is based on the powers of 10.

Rule: A number is divisible by 11 when the sum of the digits in an odd position (starting from the right) and the sum of the digits in an even position have a difference of 11, zero (0), or a multiple of 11.

When the addition of the first group of addents (in odd positions) gives a sum smaller than the sum of the second group, we must add 11 to that first group because it must always be the minuend.

Conclusion of Divisibility : With Division Chart I
– Activities With the Multiples 96 Chart

1. Observe the Division Chart I with the child:
   A. Examine the vertical column below each dividend: below 18, we find 2, 3, 6, 9 : this means that 18 is divisible by all these numbers.
   B. Note those dividends, the sum of their digits giving 9 (36, 72, etc.) and the presence of 9 in the vertical column below each. Now we know that these numbers are also divisible by 3.
   C. Note numbers divisible by 5 by locating all the quotients of 5 and checking the dividends. Their last digits prove our rule.
   D. BUT looking below 30, we note that 2 and 3 are not present though our rule for 2 tells us that 30 is divisible by 2. And by taking 3 10-bars we can say it is divisible by 3. So the chart is limited.

2. How do we discover the other divisors of 30? not shown on this chart. (Display 3 ten-bars)

   A. Write:

   30 = 3 × 10 We know this as shown with the ten-bars.
   30 = 2 × 15 We know this because 30 ends in zero.
   30 = 5 × 6 We have found this divisor on the chart.

   B. 3, 2 and 5 are prime numbers: we can see that they are the prime factors of 30. (Show slips)

   C. $\boxed{2}$ $\boxed{\times}$ $\boxed{3}$ $\boxed{\times}$ $\boxed{5}$ $\boxed{=}$ $\boxed{30}$ OR $\boxed{2}$ $\boxed{\times}$ $\boxed{15}$ $\boxed{=}$ $\boxed{30}$

   D. Then, by showing the product of any combination of the prime factors (3×5 = 15, as shown), we discover that the product (the number; here 30) is divisible by any resulting product of the factors. So 30 is divisible also by 6, 10, 15.

3. Analyze Chart of 96: We know 96 is a multiple of all these numbers; this means that 96 is also divisible by all these: 96 is divisible by 12.
   I know 12 is divisible by 4.
   So 96 is also divisible by 4.

Rule: If a number is divisible by a second number, and the second by a third, then the first number is also divisible by the third.

4. The child needs many such mental exercises which relate divisibility and multiples.

Aim – To make the child understand all those divisors of a number without carrying out the division.

Age – 9 – 10