Passages of Squares to Square Root

Passages from Squares to Cubes

Passages of Cubes to Cube Roots

THE LAST PASSAGES OF THE BINOMIAL AND TRINOMIAL

Presentation #1: Products of the Binomial with the Decimal System Materials

We have already seen the square divided into the binomial and the trinomial. The child should be very familiar with this work and with that of the construction of the decanomial. NOW we are prepared to multiply binomials which DO NOT GIVE A SQUARE AS A PRODUCT.

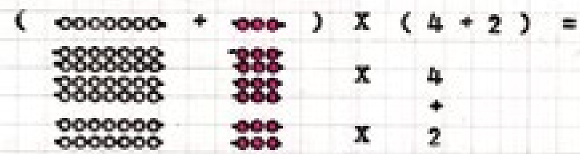
Material

1. The box of colored bead bars from 1 - 10.
2. The decimal system materials.
3. Slips and operation symbols.

1. Lay out the operation with number slips and operation signs. The child then shows the quantities of the first term with the materials.

2. The child shows 7 X 4 and 3 X 4 with the bead bars under the corresponding quantities. Then HE TURNS OVER THE 4. And begins the multiplication of the two quantities by 2.

1. This means that I have to multiply (7 + 3) first times 4 and then times 2. The addition sign between the 4 and 2 means that I have to add those two products together.



3. He puts all the bead bars together: here we have a figure other than a square: it is a rectangle.

4. Show the child how the operation is written in mathematical calculation.

$$\begin{aligned}(7+3)\times(4+2) &= \\ (7+3)\times 4 &= (7\times 4)+(3\times 4) = 28 + 12 = 40 \\ (7+3)\times 2 &= (7\times 2)+(3\times 2) = \underline{14 + 6 = 20} \\ & \qquad\qquad\qquad\qquad\quad 42 + 18 = 60\end{aligned}$$

5. Now do the reverse operation, showing the (4 + 2) as the bead quantity times the term (7 + 3). The multiplication then is by 7 and 3. The result is the same as shown by the subsequent calculation; BUT THE FIGURE WILL BE DIFFERENT.

6. The child shows the two operations simultaneously on the mat with the materials.

Presentation #2: Operations with Numbers Greater Than 10: 23 X 25.

1. We first write the proposed binomial: 23 X 25 = (20 + 3) X (20 + 5).

2. The child forms the first term with quantities (+) X .. and shows the second with symbols. NOW HE IS USING THE GOLDEN BEAD BARS, showing the 20 as two golden ten bars.

3. He begins the operation by taking 20 X 20; that is, he shows the first ten-bar X 20 and the second ten-bar X 20---he is, then, laying out 40 ten-bars, 20 below each of the originally placed ten-bars. Then he multiplies the three-bar times 20, laying out 20 three-bars below that quantity;***HE TURNS OVER THE 20, indicating that the multiplication of both quantities of the first term have been multiplied by 20. Then he multiplies each of the quantities of the first term by 5, laying out five ten-bars below each of the two shown; and 5 three-bars below the three. ***Here the child knows that 3 taken 20 times can be 20 taken 3 times SO he can show 10 X 3 with three ten bars and then 10 X 3 again with three ten bars.

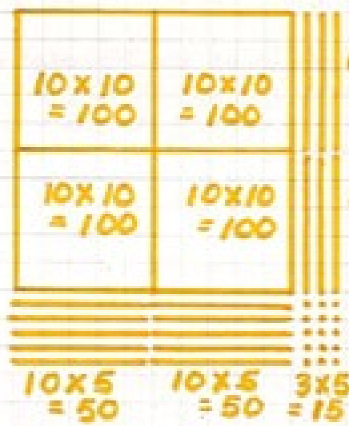
4. He combines the bead bars which show the multiplication to make a geometrical figure.

5. THEN HE WRITES THE CALCULATION: $(20+3)\times(20+5) =$

$$\begin{aligned}(20+3)\times 20 &= (20\times 20)+(3\times 20) = 400 + 60 = 460 \\ (20+3)\times 5 &= (20\times 5)+(3\times 5) = \underline{100+15 = 115} \\ & \qquad\qquad\qquad\qquad\qquad\quad 500+75 = 575\end{aligned}$$

6. Now the child can take the large square 20 X 20 and transform it with 4 100-squares. HIS CALCULATION DESCRIBES THE FIGURE.

$(20+3)\times(20+5)=$



$(20\times 20)+(3\times 20)+$

$(20\times 5)+(3\times 5)$

DIRECT AIM: The geometrical representation of products formed by units and tens.

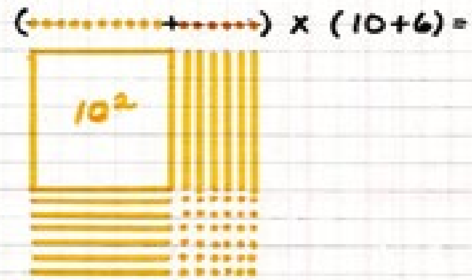
INDIRECT AIM: Preparation for the geometrical representation of multiplications done with the hierarchical materials. (colors)
NOTE: In the checkerboard work the child has a color guide to follow for the geometrical scheme; here he has abstracted further, beyond that color guide.

Presentation #3: The Binomial Square Beyond Units

1. Write on a slip 16^2. . .
then write $(10 + 6)^2$. . .

1. We want to raise to the second power 16. So we must decompose the 16 into units and tens.
$16^2 = (10 + 6)^2$

2. The child shows the operation as in the preceeding two operations. He shows the first term with the quantities in beads and the second with symbols. This means that I have to multiply (10 + 6) first by 10 and then by 6.



3. The child knows that 10 X 10 is 100, so he shows that multiplication with the 100-square. He knows that 10 X 6 is equal to 6 X 10---so he shows both of those multiplications with ten ten-bars. THEN HE SHOWS THE 6 X 6 WITH 36 UNIT BEADS.

4. He writes down what has been done:

$$\begin{aligned} 16^2 &= (10+6)\times(10+6) = \\ &= (10\times 10)+(6\times 10)+ \\ &\quad (10\times 6)+(6\times 6) \\ &= 10^2+2(10\times 6)+6^2 \\ &= 100+120+36 \\ &= 256 \end{aligned}$$

5. Clearly explain why we follow the pattern of this figure.
NOTE: When the child worked with the binomial and trinomial square he had figures like this in which he substituted the perfect squares. BUT now we are no working with linear measures that go in progressions of the powers of 10; 10 by 10. NOW we progress 100 by 100 which represents the SURFACE MEASURES. And measures of surface DO NOT FOLLOW A LINE.

5. In any multiplication, when we reach 10 of any order, we change that quantity for one of the higher order.
To follow that rule, we must change the ten ten-bars for the 100-square.
BUT we form the 6^2 with the unit beads because we have not reached 10 of the order. The square of 6 is not formed of three ten-bars and 6 units, BUT OF 36 UNITS.

THUS THIS WORK IS PARALLEL TO AREA WORK.

The Binomial Square Beyond Units . . .

6. Introduce the surface measure progression of 100 x 100 with the chart of the squares of numbers.

$1^2 = 1$	$10^2 = 100$	$100^2 = 10,000$
$2^2 = 4$	$20^2 = 400$	$200^2 = 40,000$
$3^2 = 9$	$30^2 = 900$	$300^2 = 90,000$
$4^2 = 16$	$40^2 = 1,600$	$400^2 = 160,000$
$5^2 = 25$	$50^2 = 2,500$	$500^2 = 250,000$
$6^2 = 36$	$60^2 = 3,600$	$600^2 = 360,000$
$7^2 = 49$	$70^2 = 4,900$	$700^2 = 490,000$
$8^2 = 64$	$80^2 = 6,400$	$800^2 = 640,000$
$9^2 = 81$	$90^2 = 8,100$	$900^2 = 810,000$

From the square of 1 which equals 1 to the square of 9 which is 81, we have linear measure--that is, we use the unit bead, progressing 10 by 10. BUT when we move to 10^2 we have a progression of 100---and again to the 100^2 we go by 100. The progression of the chart, then, is 100 by 100---the surface measure

ANALYZE BY COLUMNS:

Column #1: When we get to 4^2 we have 16---that is we have moved to a square formed of tens and units; but because we are raising 4 to the second power, we cannot take a ten-bar and 6 units---WE MUST TAKE 16 UNITS. And so for 5^2, 6^2, etc.

Present the squares of 1 - 9 in the cabinet of powers, noting that EACH SQUARE IS FORMED OF UNITS.

Whenever you multiply ONE NUMBER by itself, you must form a PERFECT SQUARE. So in the squares of the units from 1 - 9, no ten-bars can be used. When raising one of these numbers to the second power, we MUST FORM THE SQUARE.

Column #2: Here we note the addition of two zeros for the square of 10. When we reach 40^2, we have four digits; but the measure is actually hundreds or sixteen hundreds as noted by the two zeros. And so the measure has increased 100 by 100. With the 10^2 we begin the SURFACE MEASURE which has two dimensions and denotes every point which composes the surface. That is, we show movement in two dimensions, thus creating a surface.

Column #3: Now we have four zeros. And again the measure has increased by 100. Our measure is no longer following the line, but it follows the square. So the progression is from hundreds to ten thousands. And whenever we multiply hundreds X hundreds or hundred^2, our measure is ten thousand. SO at 400^2 we have 16 ten thousands.

NOTE: Linear measure has one dimension; surface measure has two; volume measure has three.

7. Repeat the binomial square with the decimal system materials: 14^2 $(10 + 4) \times (10 + 4)$

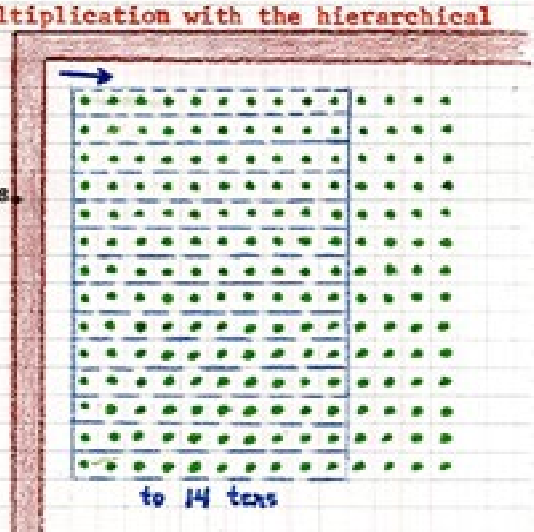
Presentation #4: The Passage from the Real Square to the Symbolic Square

DIRECT AIMS:
A. To give an understanding of the reasons why, in the squares of numbers, our progression goes 100 by 100.
B. To make the child conscious that it is impossible to carry out operations of LARGE MULTIPLICATIONS WITH NON=HIERARCHICAL MATERIALS.
C. To show the technique of reducing the real square into the hierarchical square.
D. The geometrical representation of multiplication with the hierarchical colors.

INDIRECT AIM: PREPARATION FOR THE SQUARE ROOT.

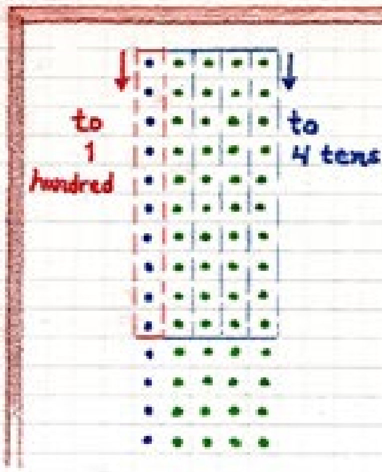
Material
1. The peg board.
2. TWO boxes of the pegs in the hierarchical colors.

1. On the board, form a large square of green pegs, 14 on a side, thus showing the square which the child has just constructed as the square of the binomial. . .with the decimal system materials.
NOTE: We build the square in this way only to show the transformation.



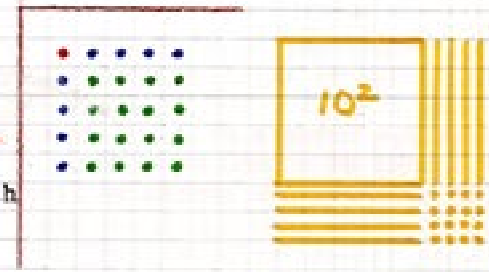
What is 14^2 equal to? 196

The Passage from the Real Square to the Symbolic Square. . .



2. Now we want to transform the green square into another square of THE SAME VALUE. BUT THIS SQUARE MUST BE SMALLER.

3. Transform the square: begin by counting the first row at the top, left to right, taking 10 green pegs and replacing with one blue. Substitute one blue for each of the ten that begin the fourteen rows. THEN move to the last four rows, counting down and substituting one ten for each ten green unit pegs. THEN transform the first long row of blue tens into one red peg.



4. THE END RESULT:
DOES IT HAVE THE SAME VALUE? Yes.
What is the value? The child reads: 196.

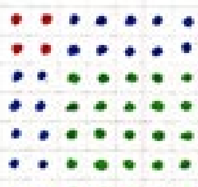
5. Now identify this square on the board with the previously formed 14^2 which the child constructed with the decimal system materials.

6. The child writes the whole operation. . . . .

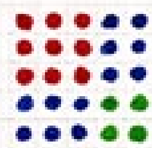
$$
\begin{aligned}
(14)^2 &= (10+4) \times (10+4) = \\
&= (10 \times 10) + (4 \times 10) + (10 \times 4) + (4 \times 4) \\
&= 10^2 + 2(10 \times 4) + 4^2 \\
&= 100 + 80 + 16 \\
&= 196
\end{aligned}
$$

7. Two more examples: THE HIERARCHICAL SQUARE AND CALCULATION.
NOTE: The child consults the table of squares for the big multiplication of the squares ($30^2 = 900$)

$$
\begin{aligned}
&25^2 \\
&= (20+5) \times (20+5) \\
&= (20 \times 20) + (5 \times 20) + \\
&\quad (20 \times 5) + (5 \times 5) \\
&= 20^2 + 2(20 \times 5) + 5^2 \\
&= 400 + 200 + 25 \\
&= 625
\end{aligned}
$$



$$
\begin{aligned}
&(32)^2 \\
&= (30+2) \times (30+2) \\
&= (30 \times 30) + (2 \times 30) + \\
&\quad (2 \times 30) + (2 \times 2) \\
&= 30^2 + 2(30 \times 2) + 2^2 \\
&= 900 + 120 + 4 \\
&= 1{,}024
\end{aligned}
$$



Presentation #1: THE ALGEBRAIC BINOMIAL: Passage from the Numerical Binomial to the Algebraic Binomial

1. Ask the child to form on the board the numerical binomial (as above); and to write the calculation. (25)2

2. Denote the two terms of the expression as a (20) and b (5). So now we must multiply both terms first by a and then by b.

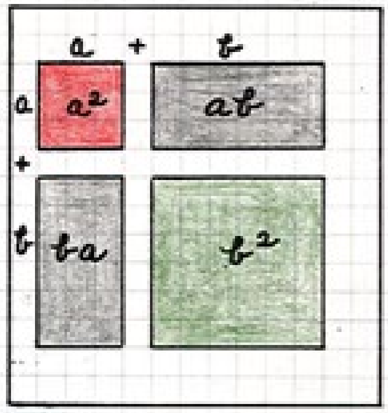
2. We are going to call 20 "a" and 5 "b." Using letters is not more difficult, but easier because instead of calculating, WE ARE ONLY INTERESTED IN SHOWING WHAT HAPPENS.

3. As the written expression develops, place the appropriate expressions written on slips on TOP of the corresponding parts of the numerical square which is shown on the board.
NOTE: We must also tell the child at this point that when we show the multiplication of two terms, we can omit the multiplication sign and still mean the same thing.

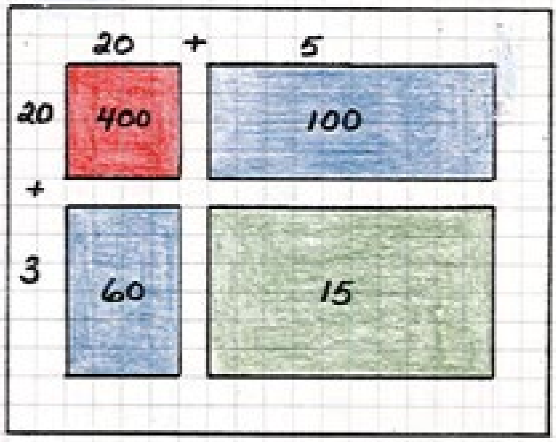
$$
\begin{aligned}
&(a+b)^2 \\
&= (a+b) \times (a+b) \\
&= (a \times a) + (b \times a) + (a \times b) + (b \times b) \\
&= a^2 + 2(ab) + b^2
\end{aligned}
$$

4. Together with the children, write the rule: A is the first term; b is the second term. In order to raise a binomial to the second power, we raise the first term to the second power PLUS two times the first term times the second term PLUS the second term raised to the second power.
NOTE: When he understands this well, he checks the rule in a math book.

5. The child can now represent his algebraic work on graph paper, using this chart as a guide:
On this chart we not longer have the blue color-coded because in algebra the colors are no longer essential.



6. The child may now do binomials which do not give a square. He may show these binomials on the board with the hierarchical pegs. AND HE MAY MAKE THE REPRESENTATIVE DESIGN, following the guide of this chart:
The only requirement for this work is that he must first decide on a unit of measure. Also he uses the color-coding.



7. At a certain point the child discovers that the units are always at the bottom of the square at the right. That the tens are always to the left side and at the top right. And that the hundreds are always found at the top left.
TAKE THE CHECKERBOARD DESIGNS WHICH THE CHILDREN HAVE DONE IN THEIR NOTEBOOKS: use as a comparison, observing the same arrangement of colors which are represented in that multiplication and this multiplication work of the binomial.

WHEN HE HAS UNDERSTOOD THE BINOMIAL WORK WELL:

Presentation #2: The Trinomial

DIRECT AIM: The geometrical representation of multiplication by means of the hierarchical arrangement, given by the rule of progression 100 by 100.

INDIRECT AIM: PREPARATION FOR THE SQUARE ROOT...and solving of second degree multiplications.

Present the written trinomial: $(235)^2 = (200 + 30 + 5) \times (200 + 30 + 5) =$
and show it decomposed, noting that first we must multiply all terms by 200, then by 30, then by 5.
Proceed through the work by showing the written calculation for the multiplication of all terms by the first (200); then show on the peg board. THEN multiply all by 30 and show that formation on the board; finally write the calculations times 5 and complete the formation.

$$
\begin{aligned}
200 \times 200 &= 40,000 \\
+\ 30 \times 200 &= 6,000 \\
5 \times 200 &= 1,000
\end{aligned}
$$

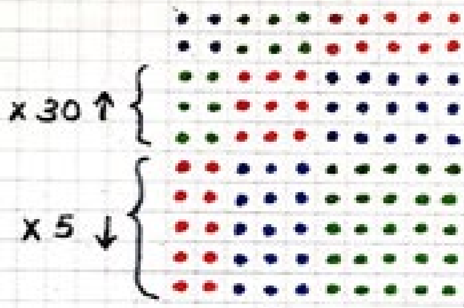
(Here we cannot use one thousand peg because we must form a figure, so we take 10 hundreds.)

The Algebraic Trinomial. . .

Now we must multiply all the terms by 30. (turn the 200 face down on the mat---that multiplication is finished)

$200 \times 30 = 6{,}000$
$30 \times 30 = 900$
$5 \times 30 = 150$

And, finally (turning the 30 down) we must multiply the terms by 5.



$(235)^2$

$200 \times 5 = 1{,}000$
$30 \times 5 = 150$
$5 \times 5 = 25$

2) AN IMPORTANT OBSERVATION : The squares are on a diagonal. The squares will always be on a diagonal because we are in the field of the measures of surface. And we have said that those measures increase 100 by 100. So we see on the diagonal the units--then the hundreds---then the ten thousands. The 100 by 100 progression. So the squares form the backbone of the square. . .and the progression is 100 by 100. We can observe that along the sides of the square we have the linear progression of 10 by 10. Moving along the right side from the bottom, we can read units, tens, and hundreds. Moving from the right along the bottom, we again have the linear progression which is units, tens and hundreds. But now we are particularly interested in that progression which is shown on the diagonal and follows the squares.

3) REVIEW: How many figures were formed in the binomial? 4
How many figures formed in the trinomial? 9
There is an arithmetical progression which corresponds to the square of the number of terms giving the number of figures formed.

4) ANALYZE each of the products represented in the figure. This calculation and the analysis of it shows the linear measure. We see that our products always increase or decrease by the powers of 10---a 10 by 10 progression. We see this in contrast to the surface measure of the diagonal squares. POINT OUT THE PRODUCTS WHICH GIVE THE SURFACE MEASURE PROGRESSION OF 100 BY 100: Units X Units always gives Units; Tens X Tens always gives Hundreds; Hundreds times Hundreds always gives Ten-thousands.

Surface measure —100— —100—

$200 \times 200 = 40{,}000$ $h \times h = M$ 4 tens of thousands
$30 \times 200 = 6{,}000$ $t \times h = k$ 6 units of thousands
$5 \times 200 = 1{,}000$ $u \times h = h$ 10 hundreds

$200 \times 30 = 6{,}000$ $h \times t = k$ 6 units of thousands
$30 \times 30 = 900$ $t \times t = h$ 9 hundreds
$5 \times 30 = 150$ $u \times t = t$ 15 tens

5) Calculate the answer by adding the products.

$200 \times 5 = 1{,}000$ $h \times u = h$ 10 hundreds
$30 \times 5 = 150$ $t \times u = t$ 15 tens
$5 \times 5 = 25$ $u \times u = u$ 25 units

$55{,}225$

10 by 10 Linear measure

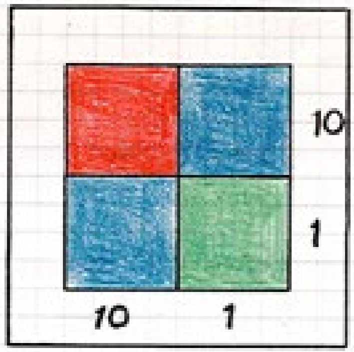
NOTE: The child is still basically working at a sensorial level. He does not yet know how the trinomial is written. When he is completely familiar with the HIERARCHICAL POSITIONS AS SEEN IN THE TRINOMIAL. . .

6) Present the Guide Charts for the Square of the Binomial and the Trinomial. In the presentation of the two charts, we simultaneously show the calculation, the resulting units as noted on the chart AND we place the pegs on the board showing that same formation as the calculation is made.

The Trinomial

Presentation of the Guide Charts for Binomial & Trinomial

$$(11)^2 = (10+1)^2$$
$$= (10+1)(10+1)$$
$$\rightarrow = (10 \times 10) = 10^2 = 100$$
$$t \times t = h$$
$$+ (1 \times 10) = \quad 10$$
$$u \times t = t$$
$$+ (10 \times 1) = \quad 10$$
$$t \times u = t$$
$$\rightarrow + (1 \times 1) = 1^2 = \underline{1}$$
$$u \times u = u \qquad 121$$



$$(11)^2 = (10+1)^2 = 10^2 + 2(10 \times 1) + 1^2 = 121$$

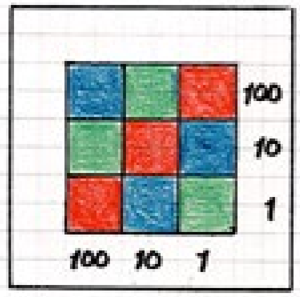
1 +	1
1	1

121

Trinomial

$$(111)^2 = (100+10+1) \times (100+10+1)$$

$= 100^2 =$	10,000	$h \times h = M$ ←
$+ 10 \times 100 =$	1,000	$t \times h = h$
$+ 1 \times 100 =$	100	$u \times h = h$
$+ 100 \times 10 =$	1,000	$h \times t = h$
$+ 10 \times 10 =$	100	$t \times t = h$ ←
$+ 1 \times 10 =$	10	$u \times t = t$
$+ 100 \times 1 =$	100	$h \times u = h$
$+ 10 \times 1 =$	10	$t \times u = t$
$+ 1^2 =$	1	$u \times u = u$ ←
	12,321	



1	1	1
1 +	1 +	1
1 +	1 +	1

12,321

The child reads the answer from the peg board display, then he adds the products to verify that answer— and, looking at the arrangement of digits, we observe with him the correspondence to the number of squares shown on the chart.

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7) Now we write the trinomial:

$$(111)^2 = (100+10+1) \times (100+10+1)$$
$$= 100^2 + (10 \times 100) + (1 \times 100)$$
$$+ (100 \times 10) + (10^2) + (1 \times 10)$$
$$+ (100 \times 1) + (10 \times 1) + 1^2$$

$h \times h = M$
$t \times t = h$
$u \times u = u$

$$= 100^2 + 2(100 \times 10) + 1^2$$

and note the diagonal of squares . . . in the written form and on the peg board. Review the hierarchies which give the hierarchies of the squares.

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8) Transforming the Numerical Trinomial Into the Algebraic Trinomial: Now we call $100 = a$, $10 = b$, $1 = c$. We are doing the same process that we have just done with numbers, but now we are only representing what happens between the three terms.

$$(111)^2 = (100+10+1)^2$$
$$= (a + b + c)^2$$
$$= (a+b+c) \times (a+b+c)$$
$$= a^2 + ba + ca$$
$$+ ab + b^2 + cb$$
$$+ ac + bc + c^2$$
$$= a^2 + 2(ab) + 2(ac) + b^2 + 2(cb) + c^2$$

How many terms do we have? 9

9) With the child, form the rule: In the trinomial we have 3 terms: a, b, and c. The square of the trinomial is equal to the square of the first term plus twice the product of the first and second terms plus twice the product of the first and third terms plus the square of the second term plus twice the product of the second and third terms plus the square of the third term.

AGE: 9 - 10

CROSS MULTIPLICATION : With the checkerboard to abstraction. . .AGE: 10 - 11

Now we take again the checkerboard for multiplication. The work which the child has done previously with the checkerboard is a strictly sensorial exercise. Now he is ready to work on a higher level. And we want to make him aware of the work which he has already done. To understand the arrangement of the checkerboard.

With the binomial and trinomial square on the peg board we begin with the 100s and the 10,000s---then gradually decrease in value as we construct the square. NOW the work of the checkerboard STARTS WITH UNITS and the squares are built. But the resulting figure is the same.

Presentation #1: $(111)^2$ on the Checkerboard : Two Ways

#1: Taking the checkerboard, we show 111 X 111 with the white numeral cards at the bottom of the board, representing the multiplicand and the grey numeral cards to the right side, representing the multiplier. Then we show on each of the squares that correspond to the usual multiplication ONE RED BEAD.

IMPORTANT is the verbalization of each multiplication:

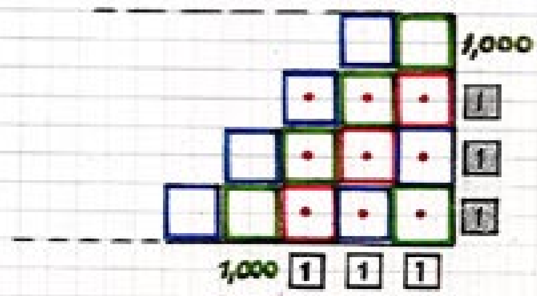
Units X units gives what? Units

Units X tens gives tens. . .

Each time we place the red unit bead as the product, we analyze the formation of the hierarchy.

NOW THE SQUARE OF THE TRINOMIAL HAS BEEN CONSTRUCTED. . .BUT WE HAVE DONE IT FROM THE UNITS UP.

We read the answer by sliding the like hierarchies down the diagonal, combining them and reading the product from the beads.



#2: Remove all the bead materials. We still show the numeral cards positioned as in the figure above. BUT now we show the products in a different order. First we multiply all terms which will give units: 1

1 X 1 = 1 We show the red bead on the green unit square.

Then all those terms which will give tens: 2

10 X 1 = 10

1 X 10 = 10 We place the two products on the blue ten squares.

Then all those terms which will give hundreds: 3

100 X 1 = 100

10 X 10 = 100

1 X 100 = 100 Show the products on the red hundred squares.

Then all those terms which will give thousands: 2

100 X 10 = 1,000

10 X 100 = 1,000 Show the products on the green thousand squares.

Then all those terms which will give ten thousands: 1

100 X 100 = 10,000 Show the product on the blue ten thousand square.

Note how many products in each hierarchy are shown: and the correspondence with the total product of the trinomial square. 12,321 We again read the answer from the materials on the combined bottom row.

Presentation #2: Real Cross Multiplication with the Checkerboard

1. Present the trinomial square: $(332)^2$. The child shows the multiplication in the usual way, using the white cards below the board as the multiplicand and the grey cards to the right side as the multiplier.

2. As in the second mode of the first presentation, the child shows the products of the multiplication by hierarchies--- first multiplying all those terms which will give units, then tens, etc. . .

2. What terms in our multiplication will give units? Then 2 X 2 = 4
What terms will give tens? tens X units.
Then we multiply 2 X 30 = 60 and
30 X 2 = 60.

When the multiplication is completed in this way for all the hierarchies, the child brings all the bars down on the diagonal and simplifies to obtain the result.

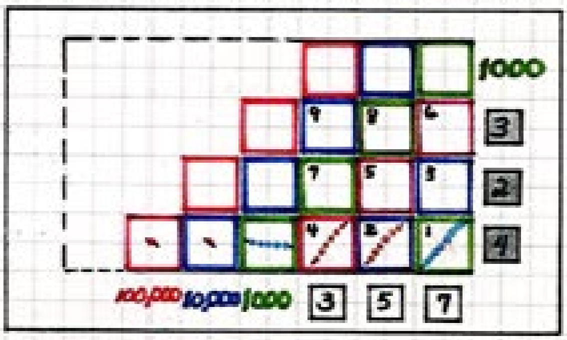
NOTE: The important thing in this work is the identification of the units of each square; and those factors which designate that particular hierarchy.

3. SECOND LEVEL: Now the child makes all the multiplications for one hierarchy and shows the sum of those multiplications with ONE BAR ON THE SQUARE CORRESPONDING TO THE HIERARCHY ON THE BOTTOM ROW. He is then, making the addition of the products for the hierarchy mentally; and he is also carrying over mentally to the next hierarchy.

3. (Hierarchy of tens)
We know that tens X units give tens.
And units X tens give tens.
So three tens times two gives 60 plus
two times three tens gives 120.
We place a two-bar on the ten square of the bottom row --- and carry over one hundred in our mind to add to the sum of the products which give our next hierarchy: hundreds.

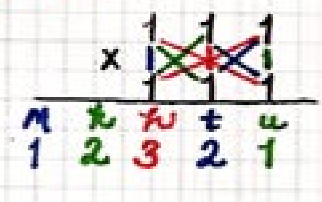
When we have finished the work, we have formed a square!!!

4. Increase the difficulty, giving the multiplication of a trinomial by a trinomial which forms a figure that IS NOT A SQUARE: 357 X 324 =
The child proceeds with the operation as described in #3. He is again doing cross multiplication: that is, all those multiplications which give one hierarchy are done as a group, the sum of those products is calculated mentally, and the sum is then shown as ONE bead bar on the corresponding square of the bottom row. And units of the next hierarchy are mentally carried over. It continues to be important in this work, that the analysis of the formation of each hierarchy is verbalized, reinforcing the combination of hierarchies which result in another.

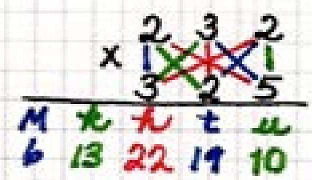


Presentation #3: Cross Multiplication Without the Material

1. The child writes the multiplication as shown, using colored lines at first to indicate his hierarchies. He begins with the simplest trinomial. $(111)^2$ A familiar one.

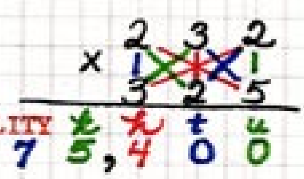


2. Increasing the difficulty, he multiplies a trinomial by a trinomial, using the cross multiplication mode which he has used with the checkerboard, but writing each partial product below the corresponding hierarchy.



3. Finally he does the same multiplication, but now he writes ONLY ONE DIGIT for each hierarchy, mentally carrying over to the next as he did in the checkerboard work.

THIS IS MENTAL FLEXIBILITY



SQUARE ROOT

All the studies of the squares that the child has done preceeds the study of the square root because it is precisely through the work of the squares that the child will be able to understand the work of the square root. This new work is the opposite of the previous work with squares. We must make sure to give him this work when he has thoroughly understood the work of the squares.

In the Montessori method, there are always three passages:

1) Experiences through activities.
2) Clarification of the points of consciousness
3) Reasoning as a consequence: the summary of the points of consciousness. This is the period when the child moves to abstraction.

When the child has carried out multiplication, he has encountered the square as a result of one number times itself. At the level of memorization, 3 X 3 forms a square. This is an indirect preparation, one the child has had since the age of 6. He has constructed the binomial square, the trinomial square, the decanomial square. When he worked with the checkerboard, he constructed the geometrical representation of multiplication: sometimes that was a square, sometimes rectangles. Again, we have the indirect preparation for square root. The child <u>must</u> have had all of these experiences.

Material

1. The squares from the cabinet of powers---from 1 - 10.
2. The board for the work of memorization of division with the green beads.
3. The decimal system material.
4. The peg board and the hierarchically colored pegs.
5. 7 bowls marked with a square indicating use for the square root: also small labels mark the different hierarchies and orders:
   - 3 bowls: white labels: green, red, blue: for units
   - 3 bowls: grey labels: green, blue, red: for thousands
   - 1 bowl: black label: green: for units of millions
6. Two guide charts for the binomial and trinomial square.
7. Two guide charts showing the result of the hierarchical square which can be applied, in its basic pattern, to the square of any number. (Shows the combination of squares and rectangles as opposed to the square pattern of $(11)^2$ and $(111)^2$ on the first two guide charts.)
8. Chart of the squares of numbers. (See binomial and trinomial work)

For the most part, this is material with which the child has already worked. Now we use it all again on another level. In the cabinet of powers, he has seen how each square is formed. . .

Presentation

1. Now we present that series of ten squares again on the mat and GIVE THE CONCEPT OF SQUARE ROOT.	1. What is this? A square. Of how many beads is it formed? 16 Each side is formed of 4 beads. The side of this square is the ROOT of the square.
2. Repeat the concept with each of the squares, emphasizing the relationship of the square to the root.	This square is formed of 36 beads. What is the root? 6 Because the side is formed of 6 beads.
3. Mental exercise: given a certain number of loose units, what square can be formed?	2. If I have 16 beads, what square can I make? A square with sides of 4 beads. If I have 81. . .
4. EMPHASIZE THE CONCEPT OF FINDING THE SQUARE ROOT OF A NUMBER BY MULTIPLYING TWO CONSECUTIVE SIDES: Note the second power. NOTE: The child MUST understand the concept of the square as a number raised to the second power because NOW we must do the opposite work to that work of the powers.	What do you do to find the square root of a number? If I have 16 beads, what square could be formed? What did you do to find out that the square had a side of 4? We MULTIPLIED 4 X 4: We multiplied two of the consecutive sides. That is, we multiplied a number times itself. When a number is multiplied times itself, we raise it to the second power.

Presentation #2: The Square Root of Units: With the Division board and beads.

1. Begin with the board and 49 beads. FORM SUCCESSIVE SQUARES FROMTHE ANGLE on the board, beginning with the square of 2, etc. until all the beads are used:

1	3	7	13	21	31	43
2	4	8	14	22	32	44
5	6	9	15	23	33	45
10	11	12	16	24	34	46
17	18	19	20	25	35	47
26	27	28	29	30	36	48
37	38	39	40	41	42	49

1. We want to know what square we can form with 49 beads and of how many beads its side will be formed.

With 49 beads we are able to form a square the side of which is 7.

But Wait - these must make a square!
WORK GUIDED BY CHART
now I can see the next BOOM! RIGHT

Squares of #s 1-9 are formed of 1 or 2 digits
" of 10-90 3 or 4
3860 →
How many digits will form the root?
12,321 ? ⟶ DIVIDE into groups of 2. h t u

2. Introduce the symbol for square root, how square root is written and NOMENCLATURE. (Include the development of the symbol.)

The square root 7, then, is written:

$\sqrt[2]{49}$

→ How many digit places the root will have — surface → linear measure

2. When we want to carry out the square root, we use a special sign called the radical sign.

This symbol began as an "r" and was written with a sweeping curve until it gave us finally this sign.

$\sqrt[2]{49}$

49 is called the radical. It is the number containing the root.

The word radical comes from the Latin word "radix" meaning "root."

3. SHOW THE CHART OF ... the number of di... columns of produ...
CONCLUSION: Because of ... 100 progression (representing the squares of area), the digits of the squares increase, colum by column, by 2.
NOTE: Therefore, the number of digits in the radical are divided into groups of two which will indicate how many digit places the root will have: a measure of surface, linear.

(Chart of Squares shown in "Binomial Square Beyond Units.")

The squares of the numbers 1 - 9 are formed of one or two digits.
From 10 - 90, of three or four digits.
From 100 - 900, of five or six digits.
SO. . .if we have the number 1600, we can determine how many digits the root will have---2.

EXERCISES WITH OTHER NUMBERS:

3,860: How many digits will form the root?
2

12,321: How many digits will form the root?
3

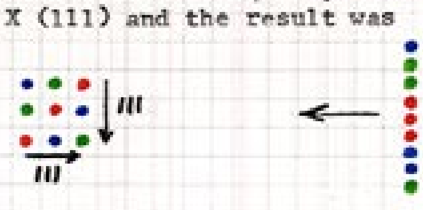
When we carry out the square root, we divide the digits into groups of 2. We can see that will indicate the number of digits in the root.

Presentation #3: From the Symbolic Square to Its Side: With t he peg board

1. Begin with the chart of the trinomial square. (Shown in "Trinomial")

1. When we constructed this square, we multiplied (111) X (111) and the result was 12,321.

2. Show 12,321 on the board with the pegs in a vertical column: then construct the square. THE WORK IS GUIDED BY THE CHART.



3. We discover where the root is read.

3. Where can we read 111? On the right side or the bottom row.
We write: $\sqrt[2]{1.23.21} \;|\; \overset{h\,t\,u}{111}$

DIRECT AIM: Direct preparation for the child to carry out square root.

Square Root. . .

Pr[...] [...]oot with the Decimal M[...]

$\sqrt[2]{6.25}$ t u

What is biggest square we can form with hundreds? Form square with side of 20

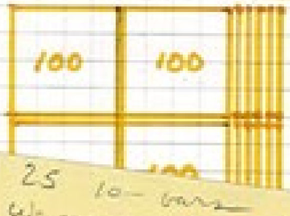
Exchange 2 squares for 20 10-bars

[...]5 in Pencil [...]l the square closed, then confirm your digit. No Remainder → Square Root is 25

1. [...] sign: mark the

1. We kn[...] the radical off by twos, [...] will be formed of two digi[...] we can also write "t" for te[...] and "u" for units.

2. [...]ares, forming



25 10-bars
We must form a square so we distribute those bars evenly on both sides.
To complete - fill in the empty square left
ROOT is 25 It works!

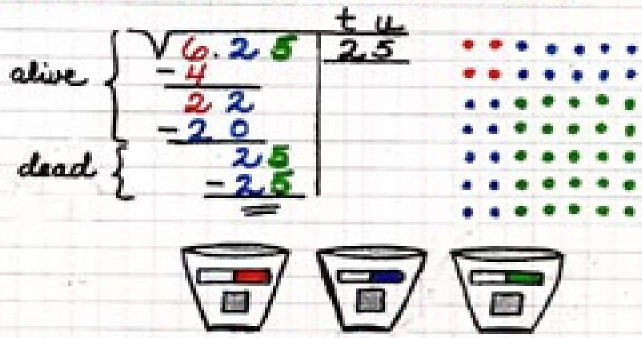
[...] of the passage [...]-successive square.

3. [...] right side or [...] ANSWER.

2. What is the biggest square we can form with hundred-squares.
We can form a square with a side of 20 using four hundred squares. Then we must stop and. . .
EXCHANGE the remaining two squares of our quantity 625 for ten-bars.
We must form a square. . .so we show those ten-bars evenly distributed on each side.
We have TWO TEN-BARS REMAINING---
EXCHANGE for units.
To complete the square, we must fill in the empty square left---we try to fill it in with these unit beads. It works.

3. The root of this square is equal to 25.

Presentation #5: Square Root with Hierarchical Materials: Second Level: Written



1. Begin by writing the square root proposition in hierarchical colors.

2. Put out the three bowls which correspond to the hierachies of the radical, and fill with the quantity of colored pegs. (Six red pegs in the hundreds bowl, two blue in the tens, five green in the units bowl.)

3. THE SAME WORK AS IN THE PREVIOUS ROOT:
   a) Form the largest possible square with the hundreds pegs on the peg board.
   b) Exchange remaining two for tens.

   a) After using the four hundreds, we can write the first digit of our square root: 2. We have formed a square the side of which is 2 (hundreds).
   b) How many hundreds did we use? 4
   How many remain? 2
   We write that subtraction.
   And exchange for twenty tens.

   c) Before distributing the tens, we bring down the ten digit. . in the written calculation.
   c) We have 22 pegs in the second bowl, so we bring down the second digit.

   d) Lay out the tens.
   d) We must form a square---we show the tens forming equal sides.

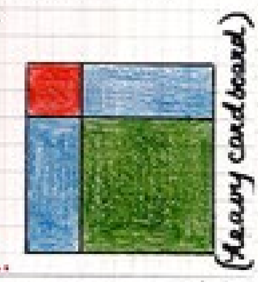
   e) Write the remainder, exchange the two remaining tens for units. Then bring down the last digit in the calculation.
   e) How many tens did we use? 5
   We can now write the second digit, but we must write it in pencil. . .only IF WE CAN CLOSE THE SQUARE WITH THE UNITS can we confirm this digit.

   f) Show the units as the final square.
   f) Now we exchange the two tens for 20 units: we have 25 pegs in the bowl. We bring down the last digit. And form the final square.

We CAN form the last square. There is no remainder. We have confirmed the last digit. We read the root. 25.

4. Show the two Square Root Guide Charts: that one used for the Trinomial and the new one which shows the combination of squares and rectangles. COMPARE: The first formed of perfect squares: the new one of squares & rectangles.

We always start with 100s → 10s → units. Here we represent any square: so the size of the interior squares & rectangles will change.



Square Root. . .
Second level. . .
Presentation #5: With hierarchical materials. . .

NOTE: In carrying out the square root, there are three moments: the first two are alive, the third is dead. In the first two we are looking for the root; in the third we have found the root. We are only filling in the last square to confirm out result. (See calculation in the presentation: marked according to the three periods.)

8. TWO PROOFS of the square root:
   a) Transcribe what figures have been formed: Analyzing the square.

$$h = t \times t = 20^2 = 400$$
$$t = t \times u = 20 \times 5 = 100$$
$$t = u \times t = 5 \times 20 = 100$$
$$u = u \times u = 5^2 = 25$$

$$400 + 100 + 100 + 25 = 625$$

   b) Multiplying one side times the consecutive side: $25^2 = 25 \times 25 = 625$

Presentation #6: The Square Root with a Remainder

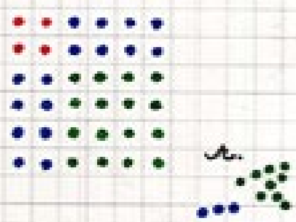
In this work, we see the importance of writing the second digit in pencil until it is confirmed in the third moment of the root extraction. We begin, as in the previous square root: 1) forming on the peg board the largest square possible with the 6 hundred pegs (looking for the first digit); 2) writing the digit, noting the 2 remaining pegs and showing that remainder in the calculation; 3) exchanging the two hundred pegs for 20 ten pegs and bringing down the second digit of the radicale; 4) adding those tens on each side of the square to form two blue rectangles (looking for the second digit; 5) writing that digit (in pencil---5), noting the one remaining blue peg and showing that in the calculation.

The first two alive moments are completed; but the third moment of confirmation shows us that the number of units, 15, is not sufficient to form the final square which must correspond to the sides of the rectangle.

SO 5 IS NOT THE RIGHT SECOND DIGIT. We must erase that digit and try 4. We must also erase the subsequent calculation. And we must take from the board two tens from each of the blue rectangles. So we now have 5 tens remaining which are exchanged (pegs) one at a time to complete the final square. With the 55 available units we are able to form the corresponding square of 16. So now the second digit is confirmed. The square is completed.

We read the square root from the peg display. The root of 615 is 24 BUT THERE IS A REMAINDER. But note that we still have three ten pegs and nine unit pegs remaining in the bowls. The subtraction in our calculation confirms a remainder of 39.

$$\begin{array}{l|l} & t\ u \\ \sqrt[2]{6.15} & 2\ \cancel{5} \\ 4 & \\ 21 & \\ \cancel{20} & \\ \cancel{15} & \end{array}$$

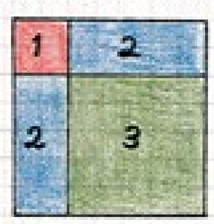


$$\begin{array}{ll|l} & & t\ u \\ & \sqrt[2]{6.15} & 2\ 4 \\ ①\{ & 4 & \\ ②\{ & 21 & \\ & 16 & \\ ③\{ & 55 & \\ & 16 & \\ & r\ 39 & \end{array}$$

Check the order of hierarchies with the guide chart. Using small numeral cards, we show on the hierarchical colors the three moments.

Review the passages:

1. Using the hundreds, we are looking for the first digit.
2. Using the tens, we are looking for the second digit of the root.
3. With the units we are confirming the second digit of the root.



ANALYZE THE PEG SQUARE:

1. The SQUARE is formed of squares and rectangles. The rectangles are determined by the SIDES OF THE SQUARES. The rectangles are the supports of the squares.
2. We will always follow the pattern: the square, then the rectangles formed on its sides.
3. Where do we read the root? The sides of the squares, found on the diagonal, will always correspond to the digits of the root. We can read the root, then, in three ways: by the sides of the squares on the diagonal, down the right side and from left to right on the bottom row. We can also read the digits of the root across the top and on the left: it will be the right quantity but not the right hierarchies.

Square Root...

Presentation #6: conclusion of the work:

The child checks the square root with hierarchical multiplications. He must add remainder.

Or he multiplies $(24)^2 = 24 \times 24$ and adds the remainder

$t \times t = h$; $20^2 =$	400
$t \times u = t$; $20 \times 4 =$	80
$u \times t = t$; $4 \times 20 =$	80
$u \times u = u$; $4^2 =$	16
$+ r =$	39
	615

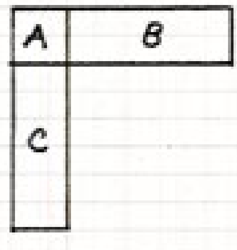
Presentation #7: A Preparation for the Square Root of the Trinomial: Reviewing Concept of Area

We know that to find the area of a rectangle we multiply b (base) $\times$ h (altitude). We have also considered in inverse rules.

That is, given A (total area) and either b or h, we can calculate the other dimension:

If $A = 28$ and $h = 4$ then $A = bh$, $28 = b \times 4$ and $b = \frac{A}{h}$, $b = 28/4$, $b = 7$

So if we have a square and we are going to build rectangles B & C on the sides of that square,



Given $A_{A+B+C} = 64\ cm^2$ and $A_A = 16\ cm^2$,

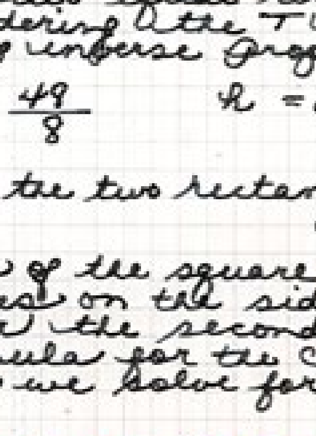
How can I obtain the dimensions of B & C?

1) Subtract 16 (Area of A) from total area. $64 - 16 = 48$
2) So 48 is the total area of B & C.
3) We know one dimension of each rectangle because it is the side of the square or 4.
4) Knowing, then, both area and b (h), we can find the second dimension of the two equal rectangles. But because we are considering the TWO rectangles, we divide by $2 \times b$, using inverse property.
5) So $h = \frac{A}{2b}$ $\quad h = \frac{48}{8}$ $\quad h = 6$
6) Therefore the dimensions of the two rectangles are (4×6).

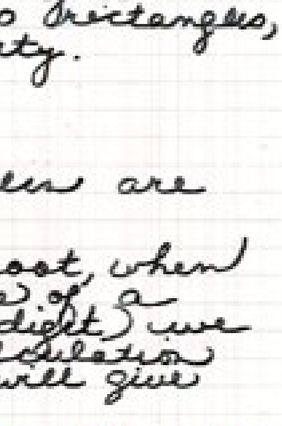
Conclusion: In the extraction of the square root, when we are building two rectangles on the sides of a square (as in the search for the second digit) we can apply the inverse formula for the calculation of area. (And the dimension we solve for will give the next digit.)

Presentation #8: The Written Square Root of the Trinomial

	z	h t u
	$\sqrt{5.56.96}$	2 3 6
①	−4	$2^2 = 4$
②	15	$15 \div 4 = 3$ (r.3)
	−12	$2(2 \times 3) = 12$
③	36	
	−9	$3^2 = 9$
④	27	$27 \div 4 = 6$ (r.3)
	−24	$2(2 \times 6) = 24$
	39	$39 \div 6 = 6$ (r.3)
⑤	36	$2(3 \times 6) = 36$
	36	$6^2 = 36$
⑥	36	



Second Guide Chart



Square Root. . .
Presentation #8: The Written Square Root of the Trinomial. . .

Analysis of the calculation and peg board design:

(1) The first passage is always the search for the first digit. We discover that the square of 2 can be built with the 5 ten thousand pegs. So the first digit is 2. We show that square to the right of the radicale calculation as $2^2 = 4$, indicating the formation of the square. . .how it was obtained.

BRING DOWN A DIGIT OF RADICALE.

(2) Now we work with 15 thousands (green pegs). We are looking for the second digit. We prolong the sides evenly, building on the side of the square. In the written calculation, we apply the inverse property and show that the second dimension of the rectangle will be 3---that is, our second digit. Then we also show the multiplication which represents the two rectangles built, how they were obtained. Our remainder is 3 thousands---they are exchanged for hundreds, ONE AT A TIME AS NEEDED. . .OR IMMEDIATELY.

BRING DOWN A DIGIT OF RADICALE.

(3) Confirmation of the second digit. The first dead passage. The formation of the square of 3 confirms our second digit AND we write $3^2 = 9$ to show how that square was obtained. Subtract in the calculation. Now 27 hundreds remain.

(4) Search for the third digit. We prolong the sides of the first square again, now the sides of the first green rectangles, looking for the two red rectangles. Again we apply the inverse property, the first dimension of the rectangles given by the side of the first square. Having built the two rectangles, we see that the inverse property is confirmed---our second dimension is 6---AND THAT IS THE TENTATIVE THIRD DIGIT. We write in pencil. We also show the multiplication for those two rectangles.

BRING DOWN A DIGIT OF RADICALE.

(5) First confirmation of the third digit: the formation of the blue rectangles, the dimensions of which are obtained from the red square and the application of the inverse property. We show again the multiplication representing the formation of the two rectangles. We know that the second dimension of that rectangle must be the side of the final square. Dead passage.

BRING DOWN A DIGIT OF RADICALE.

(6) Second confirmation of the third digit: the formation of the final square with the 36 units. We show that formation: $6^2 = 36$. Dead passage.

Read the square root as seen on the peg board. Check the square root in either mode:

1) Analysis of each hierarchy (for each geometric figure formed), writing the corresponding numerical value, and adding the results of those multiplications.
2) Multiplication of the root times itself; that is $(\text{root})^2$ to confirm the square.

Activity: The child reproduces his work with the materials on colored paper, cutting and pasting; or as a drawn design.

NOTES: On the guide chart, we show again the numeral cards which indicate the passages. We may also use two black strips to indicate the division of the "alive" and "dead" parts. And we NOTE FROM THE CALCULATION :

1) Looking for the first digit.
2) Looking for the second digit: we are using only the first digit of the second group of (two) digits.
3) With the second digit of the second group, we are confirming the second digit.
4) Search for the third digit.
5) Confirming the third digit: using the first digit of the third group.
6) Second confirmation of the third digit: using second digit of the third group

1	2	4
2	3	5
4	5	6

NOTE: A. The first square (left corner) gives one dimension to the whole top row.

B. The rectangles of simple tens have dimensions corresponding to both the red and green squares: AND these sides of the blue (last) rectangles correspond to the last two digits.

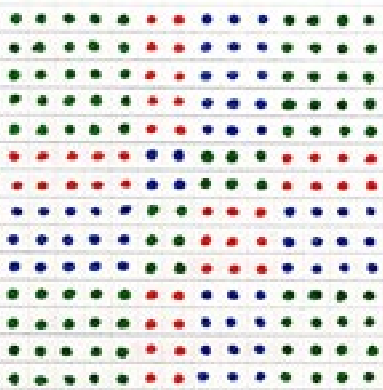
C. The first square (top left) has the greatest value: the opposite corner square (green) has the least value.

D. The squares, placed diagonally, divide the whole square into two symmetrical parts.

Presentation #9: Quadranomial Extraction of the Square Root

$\sqrt[2]{27.39.47.59}$ = 5 2 3 4 (k h t u)

1)	25	$5^2 = 25$ (1) Research of first digit
2)	23	$23 \div 10 = 2$ (2) Research of second digit
	20	$(5 \cdot 2) \cdot 2 = 20$
3)	39	
	4	$2^2 = 4$ (3) Confirmation of second digit (filling in space made by two rectangles)
4)	35	$35 \div 10 = 3$ (4) Research of third digit
	30	$(3 \cdot 5) \cdot 2 = 30$
5)	54	
	12	$(2 \cdot 3) \cdot 2 = 12$ (5) First confirmation of third digit
6)	42	$42 \div 10 = 4$ (6) Research of 4th digit.
	40	$(5 \cdot 4) \cdot 2 = 40$
7)	27	
	9	$3^2 = 9$ (7) Second confirmation of third digit
8)	18	
	16	$(2 \cdot 4) \cdot 2 = 16$ (8) First confirmation of fourth digit
9)	25	
	24	$(3 \cdot 4) \cdot 2 = 24$ (9) Second confirmation of fourth digit
10)	19	
	16	$4^2 = 16$ (10) Third (last) confirmation of 4th digit.
	3	



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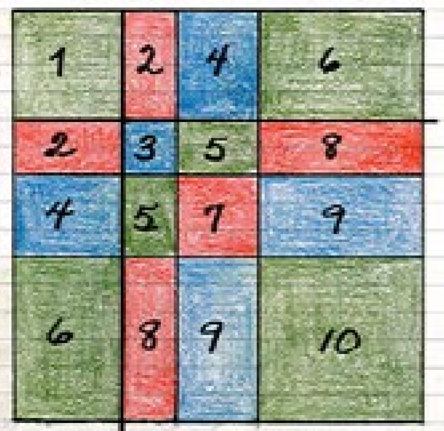


Third Guide Chart

Hierarchical check: OR Root² + remainder

k × k = m ;	$5{,}000^2$ =	25,000,000
h × k = H ;	200 × 5,000 =	1,000,000
t × k = T ;	30 × 5,000 =	150,000
u × k = k ;	4 × 5,000 =	20,000
k × h = H ;	5,000 × 200 =	1,000,000
h × h = T ;	200^2 =	40,000
t × h = k ;	30 × 200 =	6,000
u × h = h ;	4 × 200 =	800
k × t = T ;	5,000 × 30 =	150,000
h × t = k ;	200 × 30 =	6,000
t × t = h ;	30^2 =	900
u × t = t ;	4 × 30 =	120
k × u = k ;	5,000 × 4 =	20,000
h × u = h ;	200 × 4 =	800
t × u = t ;	30 × 4 =	120
u × u = u ;	4^2 =	16
		27,394,756
	+ 3	3
		27,394,759

$(5234)^2$

5 2 3 4
5 2 3 4

27,394,756
+ 3
27,394,759

The child makes one of the two checks ... and also Designs, drawing and coloring ... or with colored paper.

Presentation #9: Quadranomial Extraction of the Square Root. . . .

Materials: Add a Bowl: for the ten millions---black/blue

The child proceeds in the extraction of the square root in the same manner as for the trinomial. He first shows the quantity of the radicale in the series of bowls, using the corresponding hierarchical colors of pegs in each. Then he simultaneously shows the calculation and forms the root extraction with the pegs on the board. We analyze the steps of the work:

1) We must always consider the WHOLE FIRST GROUP. That is, when there are two digits, we use both those digits, forming the Biggest Possible Square in order to find the first digit of the root. Here our work is with the green pegs of millions. (And so the first step is actually the exchanging of the two ten million pegs for twenty millions.)

2) Research of the second digit. With hundred thousand pegs. Using the first digit of the second group.

3) Confirmation of the second digit. Second digit of the second group.

4) Still using the ten thousands. Research of the third digit.

5) EXCHANGE in the pegs is necessary to go to thousands. First confirmation of the third digit. In this work each hierarchy must be completed before going on to the next.

6) Research of the fourth digit. The material says it will be 4, but we do not know. LAST STEP OF THE "ALIVE" PART.

7) Second confirmation of the third digit. (Completes the square of the third digit.)

8) First confirmation of the fourth digit. Because hundreds pegs still remain we form the two rectangles, thus completing the work of this hierarchy.

9) Exchange remaining hundreds. Form the tens rectangles. Second confirmation of the fourth digit.

10) Final confirmation of the fourth digit. NOTE REMAINDER.

CONCLUSIONS: This is not a perfect square because we have a remainder.

How can we read the square root? on the diagonal. . .
right side and bottom. . .
left side and top 1,000

OBSERVATIONS:
A. In the trinomial we have one internal square, not related to the sides which give the root.
B. In the quadranomial, we have two squares and two rectangles which are internal and not related to those sides which give the root--- we have a greater area that is "dead."
C. We note those squares which give us the dimensions of the rectangles.

AGE: 9 - 10

AIM: Understanding the organization of the square root and Giving the possibility to visualize it.

Presentation #10: **Last Passage: Taking the Child to Abstraction**

All those passages which preceed this work are extremely important. The child must have thoroughly understood the organization of the root before this point. Now we carry out the actual construction of the root with the materials in a simpler process, utilizing what seems initially a more difficult written calculation. But it is a calculation which so faithfully describes the actual construction of the root that it provides a passage to the abstraction of clarity and simplicity when understood.

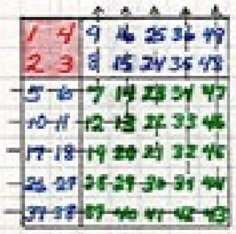
So far we have considered only one digit of the groups (as marked off in the radicand); now we consider the group of two digits together. AND now, rather than always finishing the work with the materials through one complete hierarchy before proceeding to the next, the child constructs in each period a whole square, thus working simultaneously with several hierarchies at a time. Rather than the previous work of reasoning and analysis of the resulting figures formed, the child constructs and analyzes the calculations always in terms of the square formed.

The square is now constructed from "the angle."

We begin the work with the materials by grouping the bowls containing the pegs in pairs, thus representing the digit groups.

The Binomial

	t u	
$\sqrt{6\,25}$	2 5	
-4	$2^2=4$	Largest possible square
22·5	$22 \div 4 = 5$	Bring down 25 and separate the last digit with a dot (·) This represents the confirmation digit. Now the search for the second digit proceeds and is simultaneously confirmed as the child constructs the whole square from the angle. EXCHANGE for more units as required.
-225	$45 \times 5 = 225$	
0		

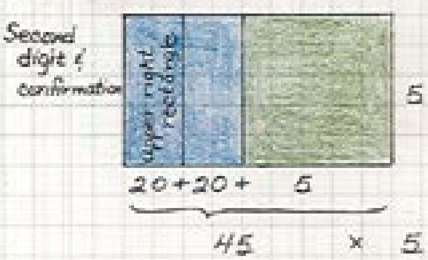


Abstractly we could divide 22 (the quantity which represents the area of the rectangles built on the sides of the square by 4. (As in previous work.) That is, because we are working on both sides of the square, we divide by the double of its side. (2+2)

The material conclusively gives the digit #2: it is 5.

How can we show that result in the calculation: we see that 225 is the result of 45 X 5. So we can formulate our rule for abstraction: To find (determine) the second digit of the root, we take the double of the first digit of our root (4) and place it in front of the quotient (which we have obtained by dividing the area of the rectangles by twice the side of the square on which they must be built); then we multiply that number times the quotient.

We finish by subtracting that product.



Understanding the rule:

a) Here we have removed the "upper right" rectangle and placed it adjacent to the "lower left" rectangle. Thus we have shown, as a large rectangle, the whole area of the square we have built (on the first square); thus the whole quantity used.

b) We can read "45" along any horizontal row of this large rectangle. And the "5," the second digit we have determined gives us the number of horizontal rows.

c) So we are calculating the area we have used in order to determine the second digit of the square root and to confirm it. That area, as shown, is 45 X 5 = 225.

The Trinomial

Write the calculation: show digits in pairs. Show the material in 3 "pairs" of bowls. The work again is done as a construction of successive squares on the angle. In this way the child discovers positively the digits of the root: the material itself sets the limit of the square. It can only be as large as the available pegs. The aim of the work, at this point, however, it to begin the movement away from the material. And so, although the digit is determined in the work with the material, the child follows the calculation through all those steps which would be necessary without the material. As he does so, he discovers the abstraction rules with a full understanding of the basis for that process of calculation.

	$\sqrt{\ }$	h t u
	21 62 25	4 6 5
1)	16	$4^2 = 16$
2)	5 62	$56 \div 8 = \cancel{7}\ 6$
	5 16	$86 \times 6 = 516$
3)	46 25	$462 \div 92 = 5$
	46 25	$925 \times 5 = 4625$

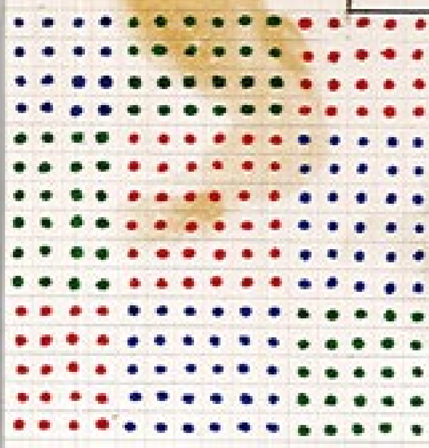
1) Forming the largest possible square.

2) Bring down second group of (two) digits. Separate the reserach of the digit from the confirmation, the dot between the 56 and the 2. With the material, an exchange is necessitated (5 ten thousand pegs for 50 thousands.) The square is formed on the angle. Exchange for hundreds as necessary. By displacing the upper right rectangle, as in the previous work, we can see the resulting area: how many we have used. 86 X 6. ABSTRACTLY we divide 56 by the double of the first digit. The quotient 7 gives us a multiplication of 87 X 7, the product of which is greater than the quantity we have available. So we try 6. And 86 X 6 works. We have the second digit.

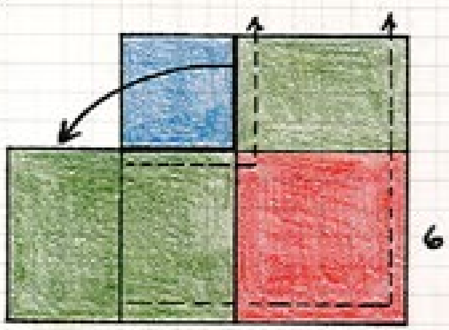
3) Search and confirmation of the third digit. We form the square from the angle. The digit, with the material is 5. Rearranging this new square, we can calculate what we have used:

$$(60 + 400 + 400 + 60 + 5) \times 5 = 4625.$$

ABSTRACTLY we divide 462 (area of the rectangles constructed on the side of the previous square which has a side now of 46) by the double of that side (92). The quotient 5 is the third digit and gives us the concluding multiplication of 925 X 5 = 4625.



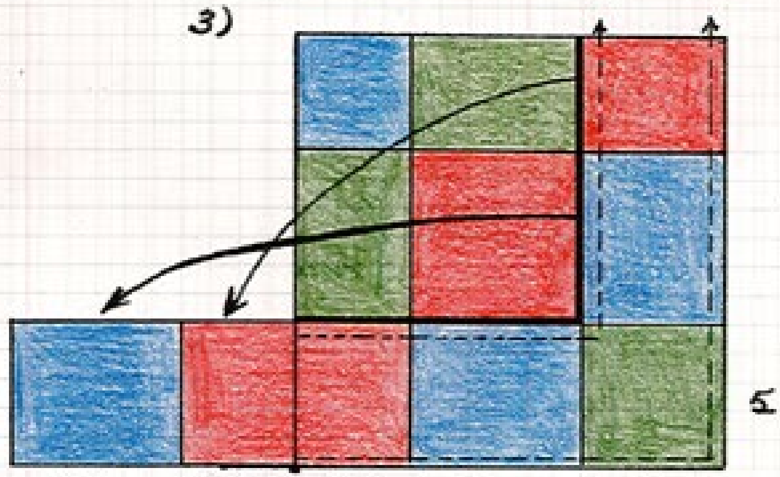
1) & 2)



40 + 40 + 6

(86 x 6)

3)



60 + 400 + 400 + 60 + 5

(60 + 400 + 400 + 60 + 5) X 5

Square Root. . .Last Passage to Abstraction
Trinomial. . .

The child, with both the work of the binomial and the trinomial in this passage, again make a proof of the square root:

1) They may make the simple multiplication of the root times itself.
2) Or show the hierarchical multiplications for each figure.
3) OR here the child may make the hierarchical multiplications representing each angle constructed; add each of those successive square sums; and then add the total of all the angles to confirm the root.

NOTE: Even when the children abandon the material, they continue often to make the drawings which represent the square root extraction, thus providing a visual representation of the extraction.

AGE: 11

DIRECT AIM: TO UNDERSTAND the abstract execution of the square root.

INDIRECT AIM: Preparation for execution of second degree equations.

SPECIAL CASES

There are three key passages which compose the work of square root extraction and preceed this work of the special cases:

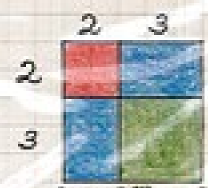
1) The concept of the root: through the squares; that is, the work with the squares from the cabinet of powers, the beads and the decimal system material.
2) Organization of the hierarchies: the passages of the different hierarchies shown in the work of construction of the root which progresses "square to rectangle to square to rectangle. . ."
3) The abstraction.

With this work, we see how important a total understanding of the square root is. The presentations of the special cases are in two periods: A) Preparation and B) An Example.

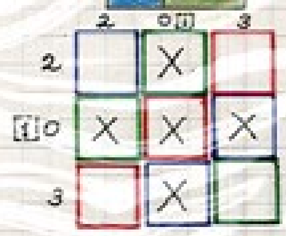
Presentation #1: The Square Root with a Second Digit of Zero: Trinomial

A) Preparation

1. Begin with the guide chart of the binomial: show with small numeral cards values for each side.

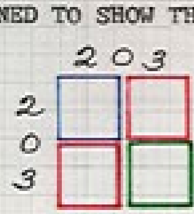


2. Take from the WHOLE SQUARES of the checkerboard the corresponding squares to construct the trinomial: organize the trinomial pattern. GIVE THE SIDES A VALUE. $(2/3)^2$



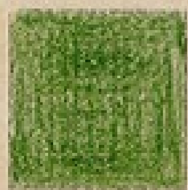
3. Substitute a zero for the second digit of the side of the trinomial.

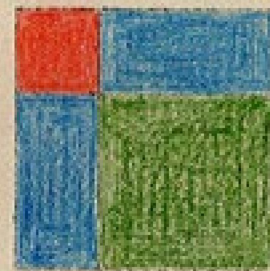
4. Turn face down those squares which represent the hierarchies which will not be present (as an actual figure in the construction.) Showing one at a time. . .why. THEN REMOVE THE SQUARES WHICH HAVE BEEN TURNED TO SHOW THE RESULTING PATTERN:

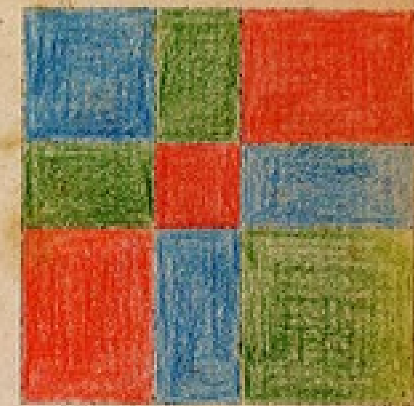


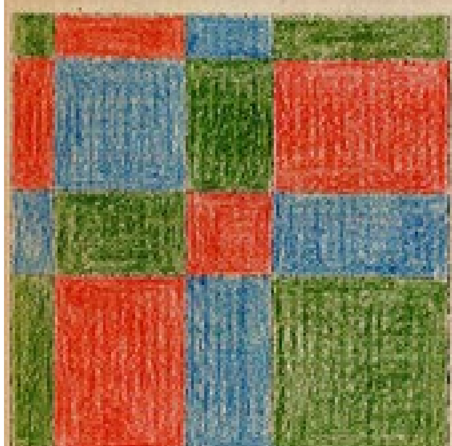
4. If we do not have units of thousands, we have no second digit---it is zero. Then we will not have the green rectangles which are formed with the units of thousands. AND we cannot have the hundreds because that hundred square is a resulting square of the rectangles. AND we will not have the two blue tens rectangles which are built on the side of that square because there is no square to support them. They must be built on the side of the square. We know, then, that we will not have these five figures when the second digit of the trinomial is zero BECAUSE OF THE ORGANIZATION OF THE HIERARCHIES.
So if we show the extraction of the square root of that trinomial with figures, this will be the resulting pattern.

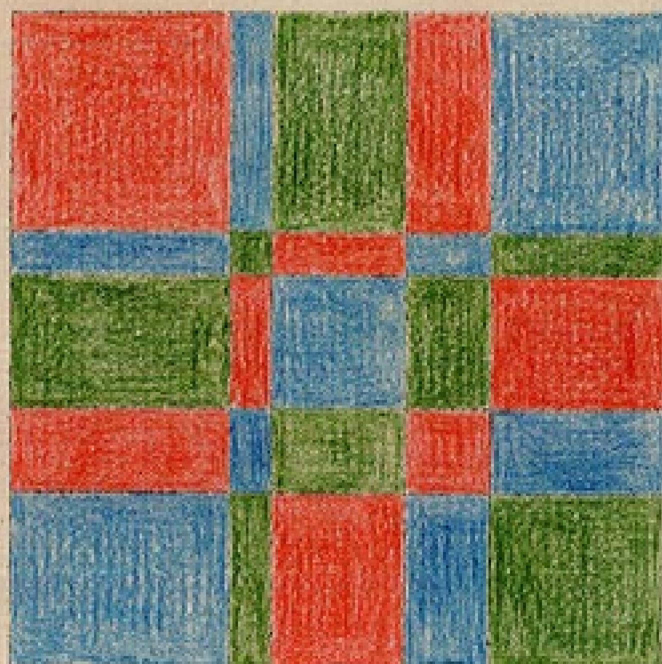
Square Root Hierarchy Guide Patterns

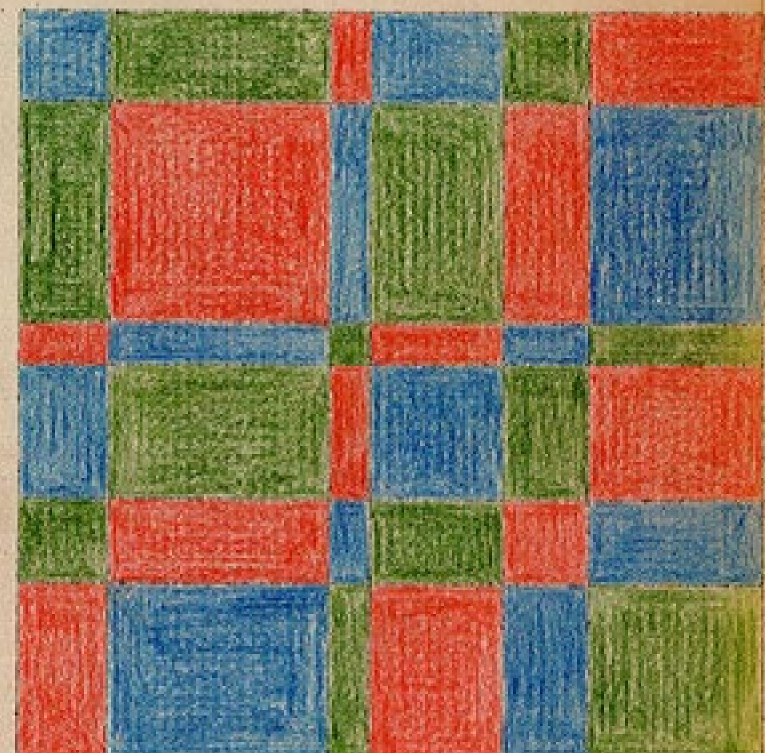

units


binomial


trinomial


quadranomial


quintanomial


sexenomial

SHEET #2

5. Show the two guide charts: of the binomial and the trinomial. Compare the new hierarchy pattern with both. Emphasize that although the missing hierarchies in the new pattern have become zero which means that it is not present in the pattern, it DOES EXIST. In our later construction, then, we see that we must indicate this.

5. Comparing this new pattern with that of the binomial:
   1) This one starts with 10,000s rather than 100s.
   2) Where we have hundreds, in the binomial we have tens.
   3) Only the units squares correspond.
   4) In the new figure the tens are missing.

   Comparing this new pattern with the hierarchy pattern of the trinomial,
   1) The 10,000 squares correspond.
   2) In the new figure the five INTERNAL FIGURES ARE MISSING.
   3) The units position is maintained.

B) Example: Figure constructed by the angle.

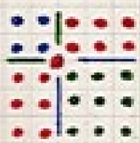
1. Simultaneously construction of the figures and calculation.

	$\sqrt[2]{4\ 12\ 09}$	h t u: 2 0 3
1)	-4	$2^2 = 4$
2)	$0\ 12$ $-\ 00$	$1 \div 4 = 0$ $40 \times 0 = 0$
3)	$12\ 09$ $12\ 09$	$120 \div 40 = 3$ $403 \times 3 = 1209$

1) Largest possible square

2) There is no remainder; we bring down the next group, but only the first digit is available to construct the rectangles on the sides of the square. We have only 1. We can't prolong the sides. THE SECOND DIGIT IS ZERO.
In the figure we must show that the hierarchies exist although we have no pegs because they are not actually present. USE THIN STRIPS OF GREEN PAPER TO INDICATE THE UNITS OF THOUSANDS. . . AND A TINY DOT OF RED PAPER TO INDICATE THE SQUAR
The placement is made as though the angle were being constructed: green strip, red speck, green strip.

3) The remainder, then, is 1200; bring down the third group of digits. Bring down the last pair of bowls: in the bowls we now have Hundreds, No Tens, and Units. Constructing with this material on the angle, we discover that we must symbolically represent the tens hierarchies which are not present: USE THIN BLUE STRIPS AS THE ANGLE CONSTRUCTION PROCEEDS---one for each of the "tens figures." The third digit is 3. ABSTRACTLY we show the division of the area for the figures built on the side of the square, which is now 40. (In the figure this is indicated by the square of 4 and the green strip representing the hierarchy which is zero.)

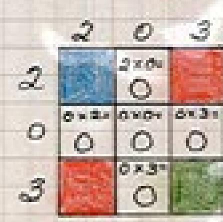


From the constructed figure, we can read the digits of the square root in the three usual ways, remembering that the presence of the paper strips indicates a zero as a digit.

2. The child makes a comparison of the resulting pattern he has shown on the peg board and the two charts of the hierarchy guide patterns: SEE PATTERNS SHEET...#2

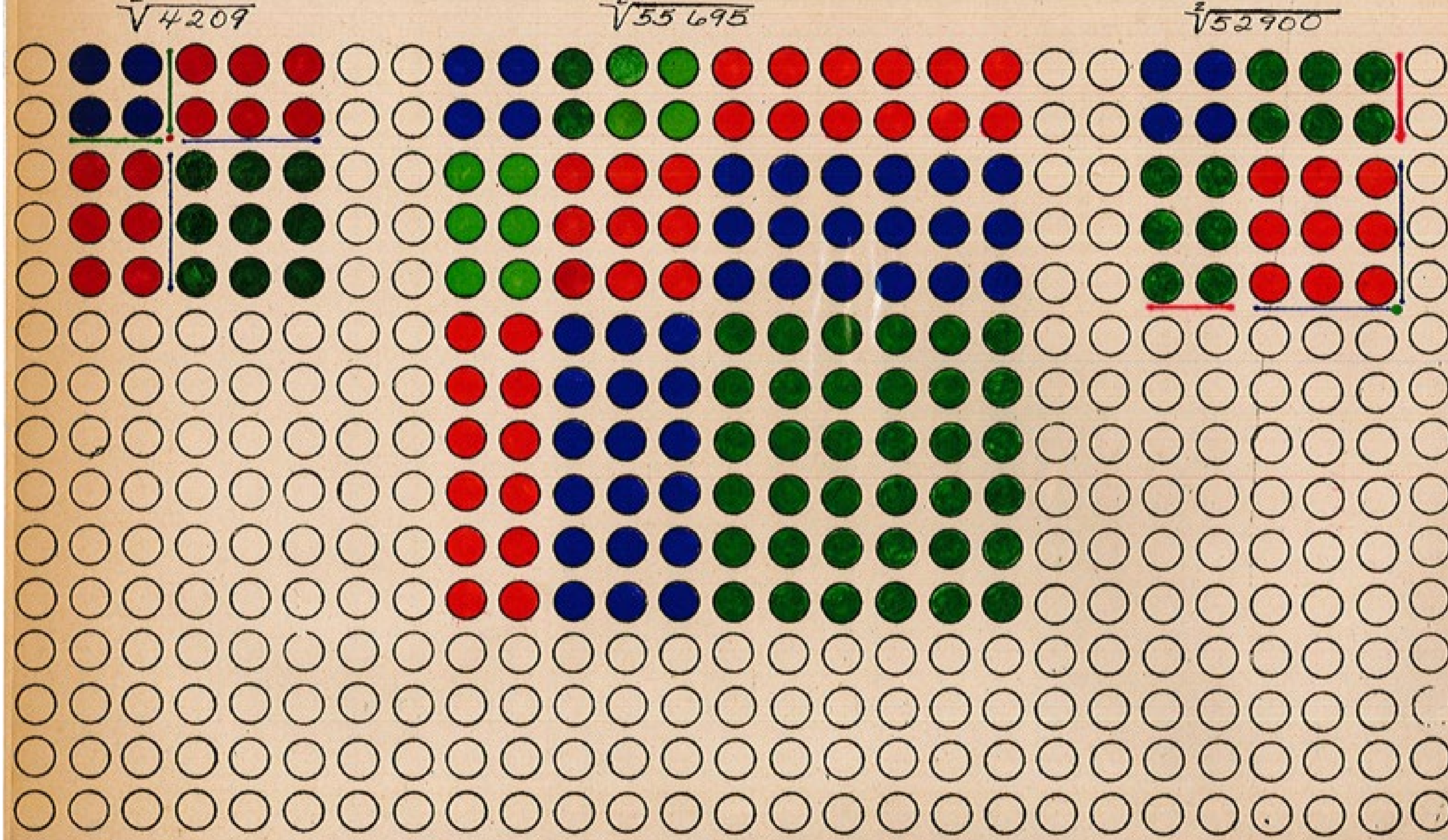
3. He draws the new and old hierarchy patterns, using the circle paper. . .SHEET #3, indicating the "Trinomial Pattern" and the "Special Case Pattern." In the second, he must draw lines which correspond to the paper strips used. A good exercise of comparison.

4. PROOF: Here the child begins by reconstructing the pattern of the trinomial with the squares from the checkerboard. He shows the numeral cards, and LOCATES THE ZEROS BY MAKING THE MULTIPLICATIONS. Again he turns over those hierarchies which the multiplications indicate as zero. THEN HE PROCEEDS WITH THE HIERARCHICAL MULTIPLICATIONS WHICH WILL GIVE THE PROOF OF HIS ROOT---ACCORDING THE THE ACTUAL FIGURES IN HIS SQUARE.



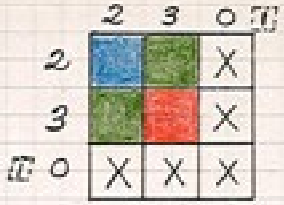
$h \times h = T;\ 200^2 = 40{,}000$
$u \times h = h;\ 3 \times 200 = 600$
$h \times u = h;\ 200 \times 3 = 600$
$u \times u = u;\ 3^2 = 9$
$41{,}209$

Square Root Extraction of the Trinomial and Special Cases



Presentation #2: Trinomial Square Root with Last Digit of Zero

A) Preparation



1. Build again the trinomial hierarchy pattern with the checkerboard squares. Give the sides a symbolic value (231) and then substitute zero for the last digit.

2. TURN FACE DOWN THOSE HIERARCHICAL FIGURES WHICH WILL NOT BE PRESENT: If the third digit is missing (that is, zero); we will also lack those three figures which represent the confirmations of that digit.

3. Form the pattern for the special case with those squares which are actually present. COMPARE THE PATTERN TO THE BINOMIAL AND TRINOMIAL GUIDE PATTERNS.

4. The problem in our calculation and formation will be: How do we read the side of the square?

B) An Example: Figure constructed by the angle.

1. In the preparation of the materials, we see that the two bowls of the last two hierarchies are empty: but we show them because the hierarchies exist.

	h t u
$\sqrt[2]{5\,29\,00}$	2 3 0
1) 4	2^2
129	$12 \div 4 = 3$
2) 129	$43 \times 3 = 129$
3) 000	$00 \div 46 = 0$
00	$460 \cdot 0 = 0$

1) Largest possible square
2) Research and confirmation of the second digit shows 3. The calculation gives no remainder; and there is no material remaining.
3) There is no material with which to research the third digit. BUT WE KNOW THAT WE MUST SHOW THE EXISTING HIERARCHIES EVEN THOUGH THEY ARE NOT PRESENT: We do this with, by the angle, a narrow strip of red, one of blue, a dot of green, a strip of blue and finally one of red.

SEE SHEET #3 for figure constructed on the peg board.

2. Ask the child to read the side of the square: 2300. NOTE that we must divide that by 10 to get the digits of the root.

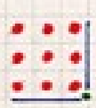
2. The right side of the square and the bottom row, where we read the digits of the root, give us 2300.
We can see by our paper strips, though, that our square does not stop here.
Our root digits myst be those hierarchies which are not present in the actual square, BUT WHICH DO STILL GIVE US THE NUMERICAL VALUE OF THE SIDE: So, because we are going to read the root digits in the next lower hierarchy, we divide 2300 by 10 = 230.

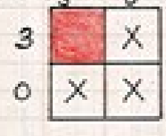
3. Child makes a PROOF in one of the two modes. He can, at this point, again reconstruct the checkerboard pattern, showing the multiplications of the hierarchies which result in zero and then proceed with the hierarchical multiplications proof. OR he multiplies the root times itself.

Presentation #3: The Binomial with zero as the second digit.

The work follows the same pattern. Here again, at the conclusion, we must read the root in a lower hierarchy, dividing 300 by 10 = 30.

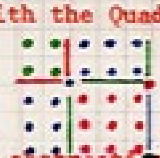
	t u
$\sqrt[2]{9\,00}$	3 0
1) 9	$3^2 = 9$
000	$00 \div 6 = 0$
00	$60 \times 0 = 0$





Presentation #4: Special Cases with the Quadranomial

	th h t u
$\sqrt[2]{4\,12\,09\,00}$	2 0 2 0



$\sqrt[2]{9\,01\,20\,04}$

AGE: 11 - 12
DIRECT AIM: Understanding of the abstraction of particular cases of the square root.

CUBE ROOT

Material

1. Box of colored bead bars.
2. Squares and cubes of all the numbers 1 - 10 from the cabinet of powers.
3. Large wooden box containing the wooden materials for raising the binomial and trinomial to the third power.
4. Cube of the hierarchical trinomial (contains the cube of the binomial.)
5. Cube of the algebraic binomial.
6. Cube of the algebraic trinomial.
7. Box of wooden neutral cubes --- 1 $cm.^3$
8. Envelope containing materials for the algebraic binomials and trinomials.

Presentation #1: From the power of a number to the power of a sum.

A. From the Number to its Square

1. Ask the child to lay out a horizontal line of the bead bars, from 1 to 10.

2. Now the child lays out bead bars corresponding to each of those shown to construct the perfect square of each number.

3. Substitute for each constructed square the real square from the cabinet of powers.

B. From the Square to its Sucessive Square

1. The child shows the construction, which he knows, of one square to its successive square. (Example: $4^2 \longrightarrow 5^2$).

2. OBSERVE with the child: what beads and bars are necessary to go from one square to its sucessive square. AND HOW THEY ARE ARRANGED: the pattern is important.

3. REVIEW THE RULE, and make certain that the child understands it: In order to pass from a square to the successive square, it is necessary to add two bead bars the length of the side of the square, one on each of two sides, PLUS ONE.

C. From the Square to a Non-successive Square

1. Ask the child to show the passage from a square to a non-successive square. (Example: $4^2 \longrightarrow 7^2$).

2. REVIEW THE RULE: To pass from a square to a non-successive square, it is necessary to add to the base square as many bars, the side of that square, as the difference between the first square and the one to which I want to pass (added on two adjacent sides of the square) PLUS THE SQUARE OF THAT DIFFERENCE.

D. From the Square to its Cube

1. Discuss the meaning of the power of a number.
NOTE: We have encouraged the child in previous experiences with the powers of the numbers to formulate these rules.

1. What is the meaning of a power of a number
The powers of numbers are like countries, some small and some large.
Some with big numerosity (populations): and some with smaller numerosity.
The numerosity of the powers is governed by perfect rules constituted of ABSOLUTE OBEDIENCE TO THE LAW OF THE GROUP.

2. THE RULES OF THE NUMBERS:
AND REVIEW THE WRITTEN NOTATION.

How do we write the powers of 7 expressing this rule???

$$7^1 = 7$$
$$7^2 = 7 \times 7$$

2. How is 7 formed?
7 is a bar formed of 7 units.
How is the square of 7 formed?
The square of 7 is formed of 7 bars of 7.
In the number 7, the rule that prevails is the number 7.
To form the number 7, we need 7 units.
To form the square of 7, we need 7 bars of 7.

3. Ask the child to form the corresponding cubes for each of the squares: he builds each cube, taking the real squares from the cabinet and NOTING that each construction obeys the law of that group. (7 squares construct the cube of 7!!)

3. When we form the cubes of the numbers, each number must correspond to its own numerosity.
1 to 1---So 1 will always be 1.
To form the cube of 2, it must obey the law of 2:

CUBE ROOT. . .
From the Square to its Cube. . .

3. . .the formation of the cubes, obeying the laws of numerosity. . .

3. . Two to the third power is the two taken twice---two times.
7 to the third power is 7 X 7 X 7:
OR 7 taken 7 times taken 7 times:
OR---we can write: $7^2 \text{ X } 7 = 7^3$.

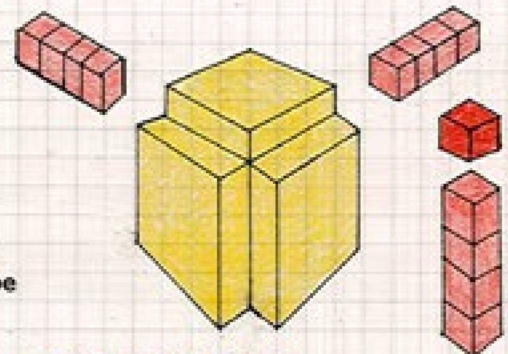
4. The child substitutes the real cubes for all the cubes he has constructed with squares.

E. Passage from a Cube to the Sucessive One: SENSORIAL

1. Introduce the box of wooden material, describing the contents: our box contains for each number: 27 squares and 1 cube. The squares are of a lighter shade, the cubes of a darker; all colors correspond to those of the bead bars. The box does not include the material for the number ten. ONLY 1 - 9. We NOTE: the squares for the squares of 1 are cubes because the dimensions for that square are 1 X 1 X 1--- the third dimension necessitated because all of the squares must have some depth dimension: one centimeter is that dimension, thus giving us the cubes of 1. Still, we can distinguish the one CUBE of one by its darker shade of red.

2. The problem: to pass from the cube of 4 to the cube of 5. Now we work with three dimensions, so we must add 3 squares:
   a) Take the cube of 4.
   b) Add the squares of 4 to THREE SIDES. We can see that we have achieved the dimension of 5 now on only one side.
   c) Add four cubes three times as shown (4 X 3).
   d) Complete the cube with the cube of 1.
   e) Take the REAL CUBE OF 5 and place it adjacent to the constructed cube to verify the construction.

2. We don't add bars of the same length because they belong to another dimension: we add squares the same length and width of the first cube.



Second Work. . . . . WRITTEN OPERATION

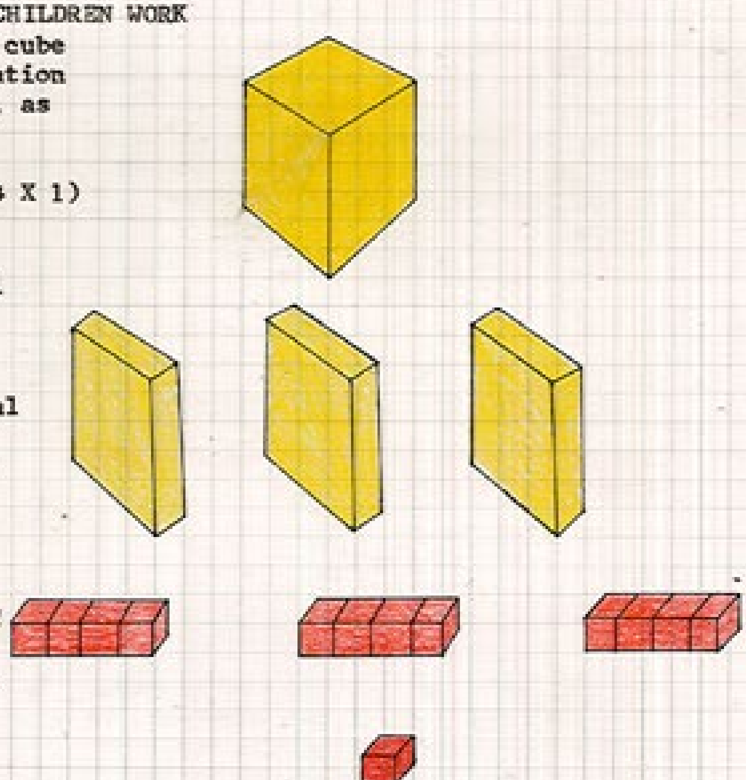
1. Dissemble the constructed cube and display as shown: THE ARRANGEMENT OF THE MATERIALS IN THIS DISPLAY IS ESPECIALLY IMPORTANT.

2. With the display as a guide, TWO CHILDREN WORK TOGETHER: One reconstructing the cube while the other writes the calculation for each step of the construction, as follows:

$$4^3 + 4^2 + 4^2 + 4^2 + (4 \text{ X } 1) + (4 \text{ X } 1) + (4 \text{ X } 1) + 1^3 =$$

$$= 64 + 16 + 16 + 16 + 4 + 4 + 4 + 1$$

$$= 125$$



3. Then the child subtracts this total from the original cube (area)--- 64, to see what has been added to form the cube of 5:

$$\begin{array}{r} 5^3 = 125 \\ -\ 4^3 = 64 \\ \hline 61 \end{array}$$ has been added!

THE CHILD DOES MANY EXERCISES PASSING FROM THE DIFFERENT CUBES TO THE SUCCESSIVE ONE.

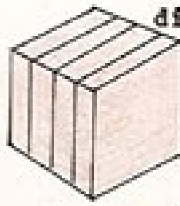
F. Passage from a Cube to the Non-Successive Cube $4^3 \rightarrow 7^3$

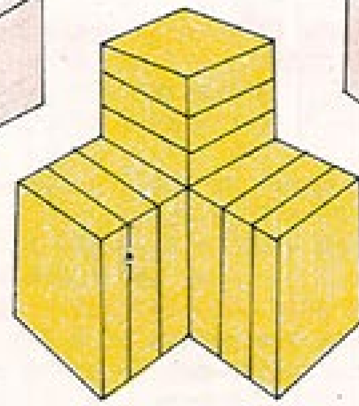
SENSORIAL

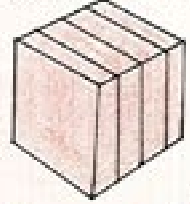
1. Review the rule of the square passage to a non-successive square.

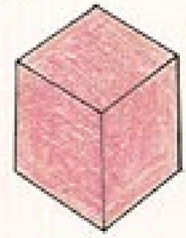
2. The problem: to pass from the cube of 4 to the cube of 7:
   a) Begin with the cube of 4.
   b) On three faces add three squares of 4 (7 - 4 = 3)
   c) We can see that we have reached an altitude of 7 on only one side. . .SO
   c) Add 4 SQUARES OF THREE-THREE TIMES, as shown.
   d) Complete the new cube with the CUBE OF THREE.
   e) Show the cube of 7 adjacent to the constructed cube to verify the construction.

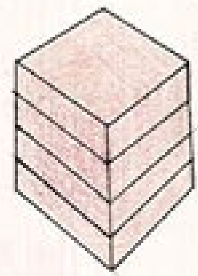
1. In the passage from a cube to a non-successive cube, we must add on three faces because we are working in three dimensions.











Second work. . WRITTEN OPERATION

1. The cube is dissembled and displayed in a pattern for analysis as in the previous passage.

	4^3	
$3 - 4^2$	$3 - 4^2$	$3 - 4^2$
$4 - 3^2$	$4 - 3^2$	$4 - 3^2$
	3^3	

2. Two children work together: Using the display as a guide, one child reconstructs the cube while the other simultaneously records the construction as a calculation--- to see what has been added for the passage.

$$7^3 = 4^3 + (4^2 \times 3) + (4^2 \times 3) + (4^2 \times 3) + (3^2 \times 4) + (3^2 \times 4) + (3^2 \times 4) + 3^3$$

$$= 64 + 48 + 48 + 48 + 36 + 36 + 36 + 27$$

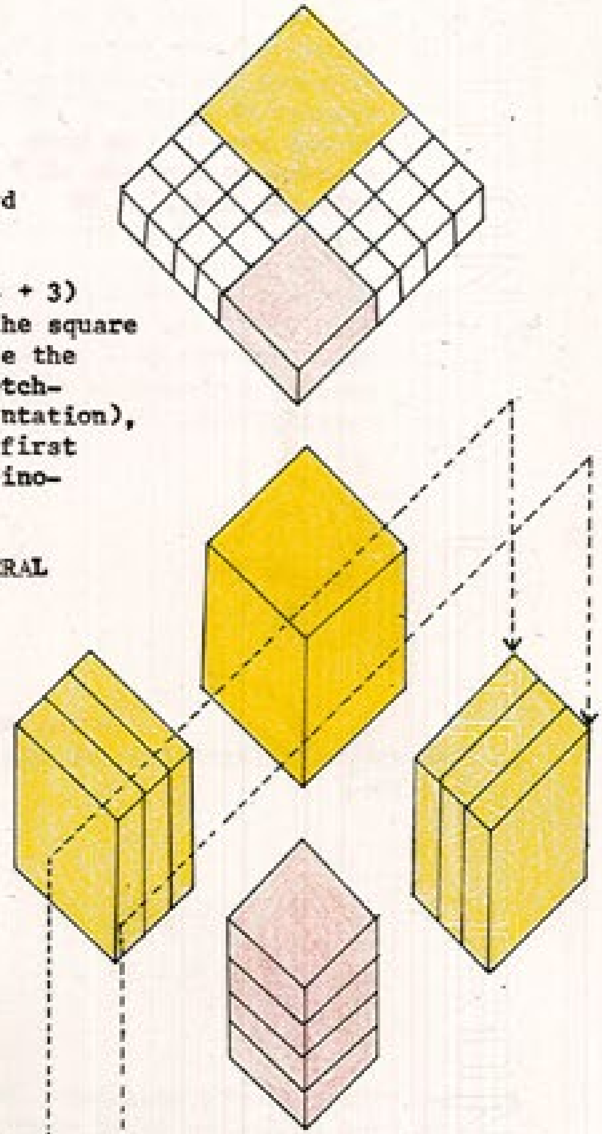
$$= 343$$

$$\begin{array}{rcr} 7^3 & = & 343 \\ -\ 4^3 & = & 64 \\ \hline & & 279 \end{array}$$ has been added

G. Cube of the Sum of Two Terms (Cube of the Binomial)

The passages from one cube to the successive and non-successive cubes take us to an understanding of the cube of a sum; that is, the cube of a binomial.

1. Review the construction of the cube in the passage from 4^3 7^3. We see that the resulting cube (7^3) corresponds to $(4 + 3)^3$. SO WE CAN WRITE THE PASSAGE $4^3 \longrightarrow 7^3$ as: $(4 + 3)^3$. BUILD THE CUBE. And that is:
$= 4^3 + 3(4^2 \times 3) + 3(3^2 \times 4) + 3^3$

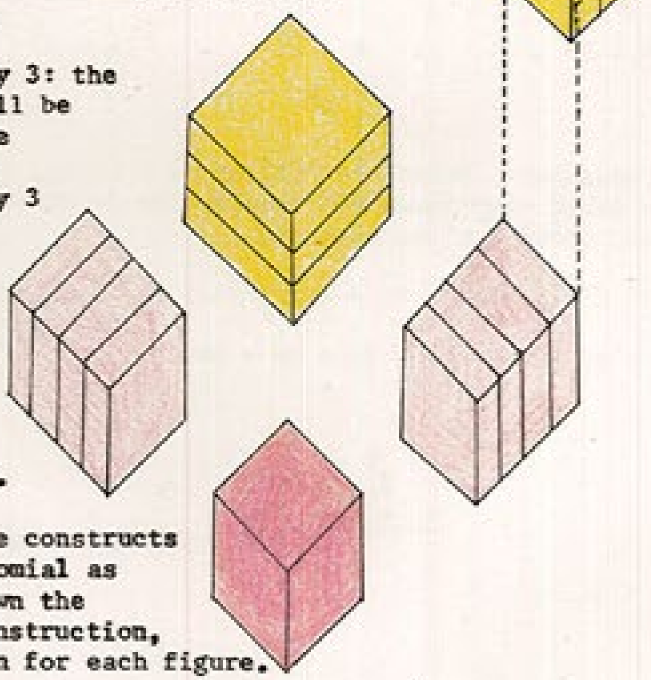


2. The problem now: To raise a binomial to the third power. IF $7^3 = 7^2 \times 7$,
THEN $(4 + 3)^3 = (4 + 3)^2 \times (4 + 3)$
$= \boxed{4^2 + 2(4 \times 3) + 3^2} \times (4 + 3)$
WE CAN SHOW THIS WITH THE MATERIAL. . . . .with the square of four, then two groups of 4 X 3 (for this we use the 1 cm^3 neutral cubes—it is handy to have them scotch-taped together as two groups of 12 for this presentation), and finally the square of 3. The result is the first figure shown. . .organized as the square of the binomial.

3. SHOW THE MULTIPLICATION OF THE BINOMIAL WITH NUMERAL SLIPS AND SIGNS ON THE MAT.

4. Now I must multiply this binomial by 4 and then by 3:
   a) Multiplying the first term by 4, we have $4^2 \times 4$—and we know that is the cube of 4.
   b) Second term—2(4 X 3)—multiplied by 4, gives us two groups of three squares of 4.
   c) Third term—$3^2 \times 4$—gives us four squares of 3.

   a) Multiplying by 3: the first term will be shown as three squares of 4.
   b) Second term by 3 gives us two groups of four squares of three.
   c) By the last term, $3^2 \times 3$ = the cube of 3.



5. The child has, as he constructs the cube of the binomial as described, also shown the notation for the construction, showing an operation for each figure.

6. Now we combine the two multiplications as represented by the figures, lifting as one piece the binomial product of 3 and placing it on top of the binomial product of 4. (indicated by dotted lines) The placement corresponds figure for figure so that we have formed the cube of 7.

7. Compare the constructed cube with the cube of 7.

8. TAKE THE ORIGINAL SQUARE (OF THE BINOMIAL) AND COMPARE IT TO EACH FACE OF THE CUBE.

Cube Root. . .

THE WORK OF THE CUBE ROOT

Presentation #1: The Concept of the Cube Root

1. Before proceeding to the understanding of the cube root, we must be sure that the child is prepared to go on. Therefore, at this point, we PRESENT AGAIN THE TWO FOLLOWING CHARTS, asking:

   What is the meaning of square root? The side of the square.
   What are the roots of the squares from 1 (1) to 81 (9)?
   Why, in looking for the square root of a number, do we divide the radicand into groups of two digits?
   Why, in looking for the cube root of a number, will we divide the radicand into groups of three digits?

1	$1^2 = 1$	$1^3 = 1$
2	$2^2 = 4$	$2^3 = 8$
3	$3^2 = 9$	$3^3 = 27$
4	$4^2 = 16$	$4^3 = 64$
5	$5^2 = 25$	$5^3 = 125$
6	$6^2 = 36$	$6^3 = 216$
7	$7^2 = 49$	$7^3 = 343$
8	$8^2 = 64$	$8^3 = 512$
9	$9^2 = 81$	$9^3 = 729$

$1^3 = 1$	$10^3 = 1,000$	$100^3 = 1,000,000$
$2^3 = 8$	$20^3 = 8,000$	$200^3 = 8,000,000$
$3^3 = 27$	$30^3 = 27,000$	$300^3 = 27,000,000$
$4^3 = 64$	$40^3 = 64,000$	$400^3 = 64,000,000$
$5^3 = 125$	$50^3 = 125,000$	$500^3 = 125,000,000$
$6^3 = 216$	$60^3 = 216,000$	$600^3 = 216,000,000$
$7^3 = 343$	$70^3 = 343,000$	$700^3 = 343,000,000$
$8^3 = 512$	$80^3 = 512,000$	$800^3 = 512,000,000$
$9^3 = 729$	$90^3 = 729,000$	$900^3 = 729,000,000$

2. Display all the cubes and corresponding squares of the numbers from 10 - 1, in that order, on the mat: give the concept of cube root.

   REVIEW THE CONCEPT OF CUBE ROOT AND SQUARE ROOT WITH ALL THE NUMBERS' MATERIAL:
   THE square root of the square is 5: we multiply 5 X 5.
   The cube root of the cube is 5: we multiply 5 X 5 X 5.

2. What is the square root of this square? (taking the 2-square) 2
   What is the square root? The side of the square.
   If the square gives the square root, the cube will give me the cube root.
   The square root of 4 (the square) is 2.
   What is the cube root of 8? (taking the cube of 2) 2
   The cube root is the edge of the cube.
   So the two roots are the same: the square root of 4 is 2; and the cube root of 8 is 2. . .because the Height of the cube corresponds to one of its sides.
   In the square we have 2 dimensions; we have 2 measures: 2 X 2.
   In the cube we have 3 dimensions and three (3) measures: 2 X 2 X 2.

3. Show how to write the cube root: we note that only the radical root number changes.

3. $\sqrt[3]{125} \;|\; \underline{5}$

4. Compare the radicale root numbers and the reasons for them:

   $5^2 \longrightarrow \sqrt[2]{25} \;|\; \underline{5}$

   $5^3 \longrightarrow \sqrt[3]{125} \;|\; \underline{5}$

   NOTE: Raising a number to a power is repeated multiplication; extracting the root is repeated division. SO. . . the cube root process for 125 is:
   $125 \div 5 \div 5 \div 5 = 1$
   Because the division by 5 is repeated three times, we have the "third root of cube root of 125. The trick is to arrive at one, an interesting comparison with the work of multiples and long (or group) division.

4. When we say a number squared, why do we use the exponent "2?"
   We write "2" because the number is repeated twice.
   When we use the exponent "3" we know that the number is repeated three times.
   The number repeated twice always forms a square.
   The number repeated three times always forms a cube.
   So when we use the number two in the radicale sign, we are indicating that number which, when repeated two times, will form a square.
   And, when we use the number three in the radicale sign, we indicate that number which, when repeated three times, will form a cube.

Cube Root. . .

Presentation #2: The Cube Root of the Binomial: With the Wooden Real Number Materials

```
            t  u
 √79 507 | 4  3
1) -64   |
   155   |
2)  48   |
   107   |
3)  48   |
    59   |
4)  48   |
   110   |
5)  36   |
    64   |
6)  36   |
    38   |
7)  36   |
     27
8)   27
     00
```

This work of the cube root is a sensorial experience, as indicated by the use of the colored "real number" materials of the cube work. We do not write any calculation, but only record in the notation what we are building.

The first step in the work is for the child to mark off the digits of the radicand into groups of three, beginning from right to left: now he knows that the root will be formed of two digits, and he indicates this with "t" and "u" The cube charts remind us of the reason for this initial process.

The cube is built from the sides and altitude, completing those dimensions in steps 1 - 4; and then it is filled in.

1) By consulting the chart of cubes, we discover that the largest possible cube contained in the first group of digits is 4: that gives us the first digit. WE BEGIN THE CUBE, THEN, WITH THE CUBE OF 4: noting that this yellow cube no longer has the value of 4 since we have tens; but we are giving a visual picture of the digit.

2) Begin research of the second digit: we know that we must add the squares of 4 as indicated by the first cube. And that we must prolong three sides here as opposed to the two sides which had to be prolonged equally in the square root.

3) Three more squares of four can be added: continuing the research of the second digit.

4) Three more squares of four can be added. With a remainder of only 11, we see that the research of the second digit is finished: we have taken three groups of (3×4^2), so our second digit is 3. (OR WE CAN COUNT SIMPLY THE NUMBER OF SQUARES WE HAVE ADDED TO ANY ONE OF THE THREE SIDES---3)

End of the alive part.

5) Now we must begin to fill in the cube. BRING DOWN ONE DIGIT. Sensorially we see that we need the squares of 3 to fill in the cube; and that we have have three groups of $(3^2 \times 4)$. (PLUS THE CUBE OF THREE) We note that we take the squares of three because three is the second digit; and 4 times because the first digit is 4. But for the child, it is simply a matter of seeing what is required to complete the cube. (3 X 3 X 4)---three times. The first group is placed and the corresponding subtraction is made.

6) Second group of $3^2 \times 4$.

7) Third group of $3^2 \times 4$.

8) BRING DOWN LAST DIGIT: Take the cube of 3 to complete the cube. The cube root--specifically the second digit---is verified.

Presentation #3: The Cube Root of the Binomial with the Hierarchical Materials

		t u	
	$\sqrt[3]{421\,875}$	7 5	
1)	343	$7^3 = 343$	Research of 1st digit
	788	$7^2 \times 3 = 49 \times 3 = 147$	
2)	735	$788 \div 147 = 5;\ 147 \times 5 = 735$	Research of 2nd digit
	537	$3(5^2 \times 7) = 3(25 \times 7) =$	
3)	525	$3 \times 175 = 525$	First confirmation of 2nd digit
	125	$5^3 = 125$	
4)	125		Second confirmation of 2nd digit
	0		

In this second passage, we record the calculation as the construction proceeds. The hierarchical materials now provide a guide for the actual hierarchical value of the digits which we discover. In the calculation, however, we continue to work with one digit at a time; and with the unit as our term of operation. The work begins with the writing of the problem; and we LAY OUT THE MATERIALS OF THE CUBE IN HIERARCHICAL ORDER, Indicating the order of construction.

1) We consult the chart of cubes to determine the largest possible cube which can be formed with 421: we see that it is 7. That gives us the first digit. It also gives a symbolic value to the side of the RED CUBE WHICH WE NOW PLACE: 70.

2) We know the three dimensions of the cube must be prolonged simultaneously: we ADD THE THREE ORANGE PRISM TO THE THREE DIMENSIONS, showing this addition of the figures in the multiplication: $7^2 \times 3 = 147$. Then, in order to determine what digit this figure represents, we must divide the total available volume of 788 by that product: 147. That gives us the second digit: 5. And the volume used: 735. Here we have subtracted all five groups at the same time.
End alive part.

3) BRING DOWN ONE DIGIT: First confirmation of the second digit. We now know the symbolic value of the next figures which must be added to the completion of the cube: $5^2 \times 7$. And we know that we must add that figure in three positions: the resulting multiplication is $3(5^2 \times 7)$ which is subtracted.

4) Second confirmation of the second digit: We add the white cube, knowing its dimensions by its symbolic value: 5^3.

Presentation #4: Moving Towards Abstraction: With the Algebraic Cube

		t u	
	$\sqrt[3]{157\,469}$	5 4	
1)	125	$5^3 = 125$	Research of 1st digit
2)	32 469	$5^2 \times 3 = 75$	
	32 464	$324 \div 75 = 4$	Research of second digit
3)	5	$3(50^2 \times 4) + 3(4^2 \times 50) + 4^3$	
		$= 30{,}000 + 2{,}400 + 64$	Confirmations
		$= 32464$	

In this passage, we bring down the whole group (of three digits). Thus, after the research of the second digit, we do the work of the confirmation using the actual hierarchical values. And THE MATERIAL OF THE ALGEBRAIC CUBE IS OUR GUIDE FOR THE CALCULATION OF THAT CONFIRMATION WORK.

1) Using the chart of cubes, we determine the largest possible cube, showing THE RED CUBE OF a^3.

2) We bring down the whole second group of three digits, and RESEARCH THE SECOND DIGIT. In order to do this, we mark off the last two digits in order to work, in the determination of the second digit, without the real hierarchical values. We now place the figures of $3(a^2b)$.

3) The confirmations now follow the pattern of the algebraic cube: we are calculating $3(a^2 \times b) + 3(b^2 \times a) + b^3$. And this acounts for the total volume which we have represented in the number 32,469. That is, we are calculating the total volume of the cube minus the volume of a^3. And so the second digit and its confirmations are represented in the HIERARCHICAL MULTIPLICATION.

Presentation #5: Carrying out the Root Abstractly (Binomial Cube)

	t u
$\sqrt[3]{205\ 579}$	5 9
125	$5^3 = 125$ Research of 1st Digit
80 5	$5^2 \times 3 = 75$
	$805 \div 75 = 9$ Research of 2nd Digit
205,579	$59^3 = 205,379$ Confirmation
− 205,379	
200	

The mode of calculation is "by the book;" and not altogether logical nor clear. But it provides the method of abstractly calculating the cube root of the binomial.

Presentation #6: The Cube Root of the Trinomial: With the "Real Number" Materials

		h t u	
	$\sqrt[3]{392\ 223\ 168}$	7 3 2	
1)	343	$7^3 = 343$	Research of 1st digit
	49 2	$7^2 \times 3 = 147$	
2)	44 1	$492 \div 147 = 3$; $147 \times 3 = 441$	Research of 2nd digit
	5 12	$3(3^2 \times 7) = 189$	
3)	1 89		First confirmation of 2nd digit
	3 23	$7^2 \times 3 = 147$	
4)	2 94	$323 \div 147 = 2$; $147 \times 2 = 294$	Research of 3rd digit
	29 2	$6(7 \times 3 \times 2) = 252$	
5)	25 2		First confirmation of 3rd digit
	41	$3^3 = 27$	
6)	27		Second confirmation of 2nd digit
	141	$3(3^2 \times 2) = 54$	
7)	54		Second confirmation of 3rd digit
	87	$3(2^2 \times 7) = 84$	
8)	84		Third confirmation of 3rd digit
	36	$3(2^2 \times 3) = 36$	
9)	36		Fourth confirmation of 3rd digit
	08	2^3	
10)	8		Fifth confirmation of 3rd digit
	0		

Although the construction of this cube root is with the "real number" wooden materials, the children must become aware of that hierarchy which corresponds to the bringing down of the next digit. Thus we display throughout this work the hierarchical cube, noting by its colors when the hierarchy has been completed, an indication of the next digit.

1) The largest possible cube, as indicated by the chart of cubes---the white cube of 7. (BLUE CUBE OF MILLIONS: a^3)

2) Research of the second digit. Through the multiplication and division in the calculation, we determine that we can add three squares of seven to each of the three dimensions: white squares. (GREEN PRISMS: a^2b)

3) Confirmation of the second digit. The construction itself indicates the need for 3^2 taken seven times on each of the three dimensions. (FAT BROWN PRISMS: $3(ab^2)$)

4) Research of the third digit. THE PRESENCE VISUALLY OF THE REMAINING BROWN PRISMS INDICATES THAT WE DO NOT BRING DOWN ANOTHER DIGIT. We determine the third digit as 2. (TALL BROWN PRISMS: $3(a^2c)$)

5) First confirmation of the third digit. Use prisms composed of the neutral cubes (7 X 3 X 2) six times. (RED PRISMS: 6(a X b X c))

6) Second confirmation of the second digit. The cube of three. (RED CUBE OF THOUSANDS: b^3)

7) BRING DOWN DIGIT. Confirmation. Cube construction indicates three groups of 2^2 taken three times. (TALL ORANGE PRISMS: HUNDREDS: $3(b^2c)$)

Second Work. . .WRITTEN OPERATION

1. The cube is dissembled and displayed in a pattern for written analysis:
   NOTE: The first construction DOES NOT INCLUDE THE TWO CUBES as shown in the previous figure.

$4 - 4^2$

$4 - 4^2$	$4 - 4^2$	$4 - 4^2$
$4 - 3^2$	$4 - 3^2$	$4 - 3^2$

$3 - 3^2$

2. Two children: One reconstructs the cube, figure by figure and the second shows the corresponding calculation:

$$4^2 + 2(4 \times 3) + 3^2 \quad \times \quad (4 + 3)$$

$$= (4^2 \times 4) + 2(4 \times 3 \times 4) + (3^2 \times 4) + (4^2 \times 3) + 2(4 \times 3 \times 3) + (3^2 \times 3)$$

3. SUBSTITUTE THE REAL CUBES FOR $(4^2 \times 4)$ and $(3^2 \times 3)$. And REWRITE:

$$= 4^3 + 2(4^2 \times 3) + (3^2 \times 4) + (4^2 \times 3) + 2(3^2 \times 4)$$

4. SUPERIMPOSE THE TWO LAYERS. And make the calculation:

$$= 64 + 96 + 36 + 48 + 72 + 27$$

$$= 343$$

$= 7^3$ COMPARE WITH THE REAL CUBE OF 7.

Third Work. . .RAISING THE SUM TO THE THIRD POWER, CONSTRUCTING FIRST THE SIDES AND THE ALTITUDE: Here we no longer begin with the base layer, but instead we construct simultaneously the three dimensions of the cube; achieving first the three sides and altitude. . .AND THEN FILLING THE CUBE IN.

$(4 + 3)^3 = (4 + 3)(4 + 3)(4 + 3)$ Dissemble the cube again, rebuilding it and showing the written operations:

$4 \times 4 \times 4 = 4^3$ . . . . The cube of 4.
$4 \times 3 \times 4 = 4^2 \times 3$ . . . . The square of 4 three times built on one side of the cube.
$3 \times 4 \times 4 = 4^2 \times 3$. . . . . Three squares of 4 built on a second side.
$4 \times 4 \times 3 = 4^2 \times 3$. . . . . Three squares of 4 built on a third side.

At this point we have two sides of the cube and its height: now we must fill it in.

$4 \times 3 \times 3 = 3^2 \times 4$. . . . . .Four squares of three to complete the altitude of one side.
$3 \times 4 \times 3 = 3^2 \times 4$. . . . . .Four squares of three to complete a second altitude.
$3 \times 3 \times 4 = 3^2 \times 4$. . . . . .Four squares of three complete two sides.
$3 \times 3 \times 3 = 3^2$ . . . . .Cube of three completes the cube.

NOTE: The multiplications represent the figures used for the construction: in the multiplication as indicated: $(4 + 3)(4 + 3)(4 + 3)$, the child may draw connected arcs below to show those terms multiplied each time.

REWRITE: $4^3 + 3(4^2 \times 3) + 3(3^2 \times 4) + 3^3$ DISSEMBLING THE CUBE ONCE AGAIN AND DISPLAYING according to this new written operation. NOTE: We see eight figures indicated in the written operation; and eight figures shown with which we construct the cube.

$2^3 = 8$ The cube of the BINOMIAL is constructed with eight figures, that is, the cube of 2. When we have a binomial raised to the third power, we will have eight figures:

AND when we raise the trinomial to the third power we will have $3^3 = 27$ figures.

H. The Sum of Three Terms Raised to the Third Power: The Cube of the Trinomial

First Work. . .SENSORIAL CONSTRUCTION

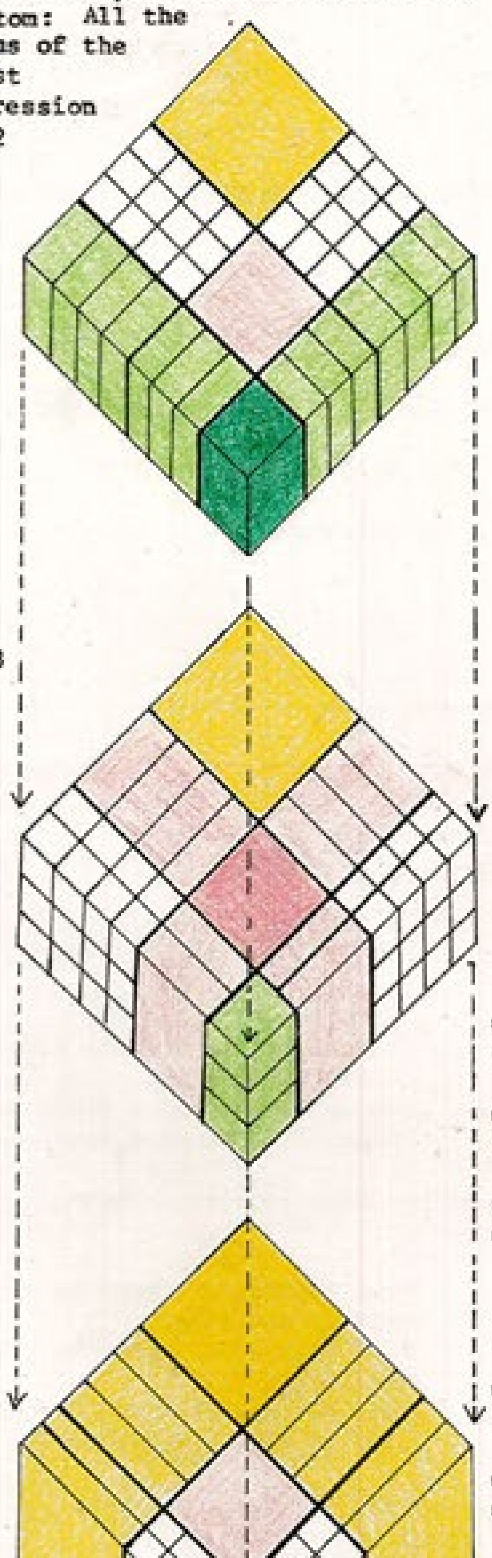
1. The problem: $(4 + 3 + 2)^3 = (4 + 3 + 2)^2 \text{ X } (4 + 3 + 2)$

THE CONSTRUCTION IS BY LAYERS, beginning with tha layer as shown here at the bottom: All the terms of the first expression

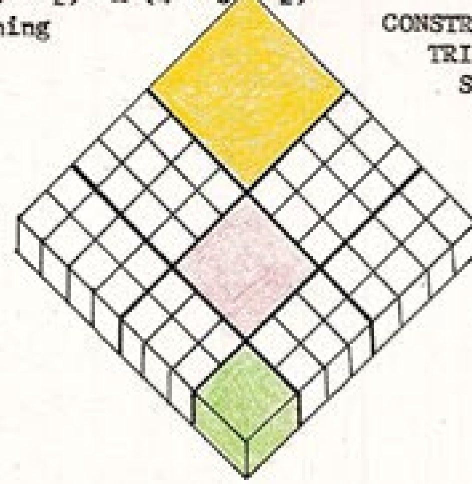
X 2

X 3

X 4



CONSTRUCT THE TRINOMIAL SQUARE:



NOTE: All those rectangles which are composed of the small neutral wooden cubes are scotched together for ease in handling.

Second Work: WRITTEN OPERATION

1. The cube is dissembled and displayed; then reconstructed, one child building and a second recording the construction with the written notation:

$(4 + 3 + 2)^2 \text{ X } (4 + 3 + 2)$
$= (4^2 \text{ X } 4) + 2(4 \text{ X } 3 \text{ X } 4) + 2(4 \text{ X } 2 \text{ X } 4)$
$+ (3^2 \text{ X } 4) + 2(3 \text{ X } 2 \text{ X } 4) + (2^2 \text{ X } 4)$

$= 4^3 + 2(4^2 \text{ X } 3) + 2(4^2 \text{ X } 2) + (3^2 \text{ X } 4)$
$+ 2(3 \text{ X } 2 \text{ X } 4) + (2^2 \text{ X } 4)$

$= 64 + 96 + 64 + 36 + 48 + 16$
$= 324$

****

$(4^2 \text{ X } 3) + 2(4 \text{ X } 3 \text{ X } 3) + 2(4 \text{ X } 2 \text{ X } 3) +$
$+ (3^2 \text{ X } 3) + 2(3 \text{ X } 2 \text{ X } 3) + (2^2 \text{ X } 3)$

$= (4^2 \text{ X } 3) + 2(4 \text{ X } 3^2) + 2(4 \text{ X } 2 \text{ X } 3)$
$+ 3^3 + 2(3^2 \text{ X } 2) + (2^2 \text{ X } 3)$

$= 48 + 72 + 48 + 9 + 36 + 12$
$= 243$

****

$(4^2 \text{ X } 2) + 2(4 \text{ X } 3 \text{ X } 2) + 2(4 \text{ X } 2 \text{ X } 2)$
$+ (3^2 \text{ X } 2) + 2(3 \text{ X } 2 \text{ X } 2) + (2^2 \text{ X } 2)$
$= (4^2 \text{ X } 2) + 2(4 \text{ X } 3 \text{ X } 2) + 2(4 \text{ X } 2^2)$
$+ (3^2 \text{ X } 2) + 2(3 \text{ X } 2^2) + (2^2 \text{ X } 2)$

$= 32 + 48 + 32 + 18 + 24 + 8$
$= 162$

****

$324 + 243 + 162$
$= 729 \quad = \quad 9^3$ (SHOW CUBE OF 9.)

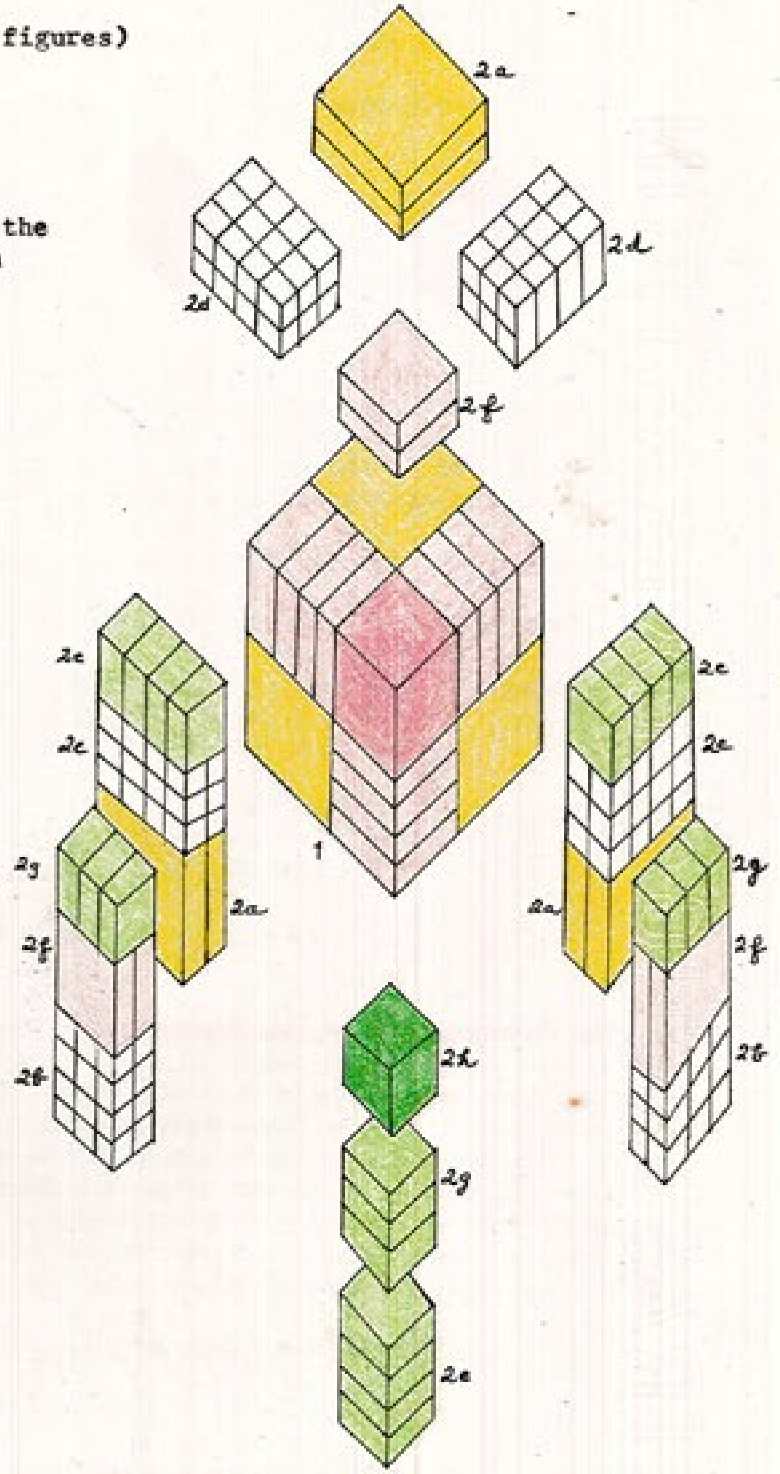
Cube Root. . .

H. The Cube of the Trinomial. . .

Third Work: Constructing the Cube of the Trinomial Through the Diagonal: Construction of the Sides and Altitude and Then Filling In.

1. Reconstruct the cube of the binomial $(4 + 3)^3$: constructing from the sides and writing again the calculation, figure by figure:

$(4 + 3)^3$
$= 4 \times 4 \times 4 = 4^3$
$+ 3 \times 4 \times 4 = 4^2 \times 3$
$+ 4 \times 3 \times 4 = 4^2 \times 3$
$+ 4 \times 4 \times 3 = 4^2 \times 3$ (NOTE: 8 figures)
$+ 3 \times 3 \times 4 = 3^2 \times 4$
$+ 4 \times 3 \times 3 = 3^2 \times 4$
$+ 3 \times 4 \times 3 = 3^2 \times 4$
$+ 3 \times 3 \times 3 = 3^3$



2. Going to the trinomial: constructing the three dimensions first as the notation indicates:

$(4 + 3 + 2)^3 =$

a) $4 \times 2 \times 4 = 4^2 \times 2$
$2 \times 4 \times 4 = 4^2 \times 2$
$4 \times 4 \times 2 = 4^2 \times 2$

b) $3 \times 2 \times 4 =$
$2 \times 3 \times 4 =$ $2(4 \times 3 \times 2)$

c) $4 \times 3 \times 2 =$
$2 \times 4 \times 3 =$ $2(4 \times 3 \times 2)$

d) $4 \times 2 \times 3 =$
$3 \times 4 \times 2 =$ $2(4 \times 3 \times 2)$

e) $4 \times 2 \times 2 = 2^2 \times 4$
$2 \times 4 \times 2 = 2^2 \times 4$
$2 \times 2 \times 4 = 2^2 \times 4$

f) $3 \times 2 \times 3 = 3^2 \times 2$
$2 \times 3 \times 3 = 3^2 \times 2$
$3 \times 3 \times 2 = 3^2 \times 2$

g) $3 \times 2 \times 2 = 2^2 \times 3$
$2 \times 3 \times 2 = 2^2 \times 3$
$2 \times 2 \times 3 = 2^2 \times 3$

h) $2 \times 2 \times 2 = 2^3$

NOTE: We have 27 figures.

Fourth Work: WRITTEN NOTATION OF THIS OPERATION

1. Dissemble the cube and display the figures horizontally in groups, representing the 27 figures:

$$4^3 \text{ -- } 3(4^2 \times 3) \text{ -- } 3(4^2 \times 2) \text{ -- } 3^3 \text{ -- } 3(3^2 \times 3) \text{ -- } 6(2 \times 3 \times 4) \text{ -- } 3(3^2 \times 2) \text{ --}$$
$$3(2^2 \times 4) \text{ -- } 3(2^2 \times 3) \text{ -- } 2^3$$

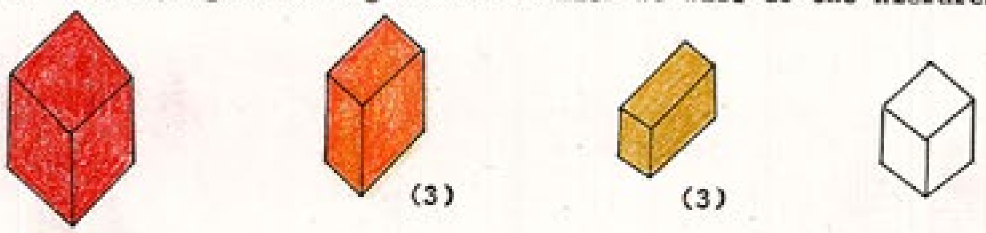
2. Reconstruct the cube from the sides as before, going from the displayed groups and making the notation:

$$(4+3+2)^3 = 4^3 + 3(4^2 \times 3) + 3(4^2 \times 2) + 3^3 + 3(3^2 \times 4) + 6(4 \times 3 \times 2) + 3(4^2 \times 2) + 3(2^2 \times 4) + 3(2^2 \times 4) + 3(2^2 \times 3) + 2^3$$

J. The Hierarchical Cube of the Binomial

We have raised the binomial and the trinomial to the third power when those expressions were composed of units. Now we must raise both to the third power with hierarchical values. For this we take the materials for the hierarchical cube of the binomial from the hierarchical trinomial box. The materials are the figures which compose the cube with values of units, tens, hundreds.

1. Begin by displaying those figures with which we will do the hierarchical work:



We note that we no longer have separate pieces: we have two cubes and other figures which represent those groups of squares which we have used before in the cube of the binomial. (SHOW A COMPARISON, taking the group of three four-squares used in the construction of the binomial as the first level and comparing it with the orange rectangular prism.) Juxtapose the two cubes.

2. Give the hierarchical value for each piece, writing the notation:

Red cube:	10^3	$= 10 \times 10 \times 10$	$= t \times t \times t$	$= k$	$= 1{,}000$
Orange :	$10^2 \times 1$	$= 10 \times 10 \times 1$	$= t \times t \times u = h =$		100
	$10^2 \times 1$	$= 10 \times 10 \times 1$	$= t \times t \times u = h =$		100
	$10^2 \times 1$	$= 10 \times 10 \times 1$	$= t \times t \times u = h =$		100
Tan :	$1^2 \times 10$	$= 1 \times 1 \times 10$	$= u \times u \times t =$		10
	$1^2 \times 10$	$= 1 \times 1 \times 10$	$= u \times u \times t =$		10
	$1^2 \times 10$	$= 1 \times 1 \times 10$	$= u \times u \times t =$		10
White:	1^3	$= 1 \times 1 \times 1$	$= u \times u \times u =$		1
					1,331

J. The Construction of the Hierarchical Binomial Cube

The problem: To raise 54 to the third power. Now we do not use the first passages, but we begin with the cube ($10^2 \times 10$) which will here be given the value of $(50)^3$ We have already said that this first cube is equal to units of thousands; so we can begin with the whole cube. In the same way, we can take the orange prism and give it a symbolic value of ($50^2 \times 4$) because we have so designated the figure as ($t \times t \times u$) hundreds. When we establish the problem, then, we are giving the sides of the hierarchical figures certain symbolic values. In the work, we construct the cube and make the corresponding notation.

$$54^3 = (50+4)^3 = (50+4)(50+4)(50+4)$$
$$= 50^3 + 3(50^2 + 4) + 3(4^2 \times 50) + 4^3$$
$$= 125{,}000 + 30{,}000 + 2{,}400 + 64$$
$$= 157{,}464$$

K. Introduction of the Chart of the Cubes of Numbers 1 - 9 (10 - 90), (100 - 900)

With the chart of the cubes of numbers, we are demonstrating the progression of the number of digits according to the powers of the numbers: we see an increase of three digits between the cube of 1 and the cube of 10---and again an increase of three digits between the cube of 10 and the cube of 100. That is, with the addition of one zero in the cube root, we have an increase of three digits in the radicand. This is in contrast to the progression by twos in the radicand of the square root. And so, with this chart, we show why, in the cube root work, the radicand is divided from right to left into groups of 3 (instead of the groups of two for the square root.

Specifically, we note in the first column the number of digits in the cubes of the numbers 1 - 9: 1 - 3; second column cubes of the numbers to the second power, 4 - 6; third column cubes of the numbers to the third, 7 - 9.

We observe that the number of digits representing the hierarchical cube is 6: for the cube of 50 (red); and 2 for the cube of 4 (white.) $(54)^3$

SEE CUBE ROOT I for the complete chart.

L. Raising the Binomial to the Third Power with an Algebraic Value

For this work we take the ALGEBRAIC CUBE (the figures which compose it shown below.) And we give the binomial an algebraic value. Prior work with "letter operations" is indicated.

First Work. . .SENSORIAL CONSTRUCTION

1. Present the problem with small numeral symbols and signs on the mat.

$$(a^2 + \begin{smallmatrix} ab \\ ab \end{smallmatrix} + b^2) \times (a + b) = (a + b)^3$$

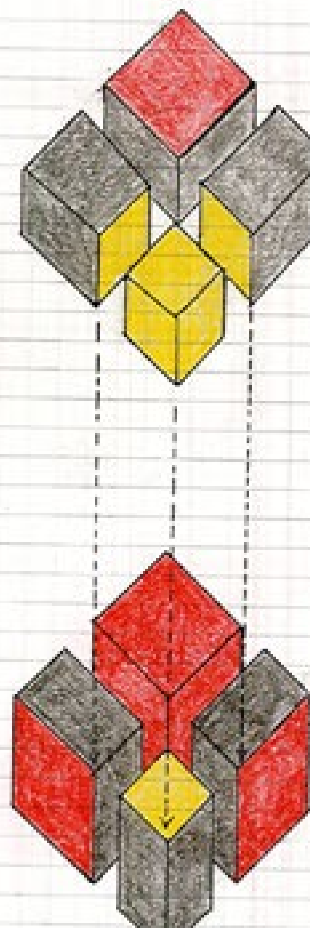
2. Stating the problem: To raise a binomial to the third power we must multiply all the terms within the parentheses times "a" and then times "b."

3. Show on the mat a prepared "base guide card" that corresponds to the square of 11 (as shown). . .on which the materials of the algebraic cube fit exactly. This card provides a guide for the construction of the cube; and one is used for both periods of the construction: X a. . .and X b.

4. Each term is multiplied by a: verbally we describe each figure in "letter terminology:"
"What does "a^2" mean? SHOW ONLY THE FACE OF THE CUBE.
Then how do we multiply it times a? SHOW THIRD DIMENSION OF CUBE.
And PLACE ON CARD. . .

NOTE: During the multiplication times "a" we show the numeral cards: " X a" beside the base card, bringing the multiplied term for each operation alongside those cards, then returning it to the parentheses.

When all terms have been multiplied by a, we return a to the equation, turn it over, bring " X b" beside a SECOND BASE CARD and construct the second layer.



X b

X a

X a

$a^2 \times a = a^3$	$ab \times a = a^2b$
$ab \times a = a^2b$	$b^2 \times a = ab^2$

X b

$a^2 \times b = a^2b$	$ab \times b = ab^2$
$ab \times b = ab^2$	$b^2 \times b = b^3$

5. Unite the layers to form the cube: We have constructed the cube of (a + b)

Cube Root. . .

L. Binomial to the Third Power with the Algebraic Materials. . .

Second Work. . .WRITING THE OPERATION

1. Display the parts of the algebraic cube as indicated below. IDENTIFY EACH WITH THE CORRESPONDING LETTER CARD placed on top of each figure:

a^3 a^2b a^2b a^2b ab^2 ab^2 ab^2 b^3

2. Combine the labels in groups---and the figures in groups to show:

	a^2b	ab^2	
a^3	a^2b	ab^2	b^3
	a^2b	ab^2	

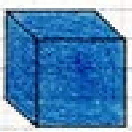
3. Introduce new numeral (letter) cards to show:

a^3 $3(a^2b)$ $3(ab^2)$ b^3

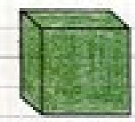
M. The Cube of the Trinomial of Hierarchical Value: With the Hierarchical Cube

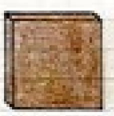
1. Display the figures of the hierarchical cube of the trinomial horizontally on the mat, according to hierarchies (beginning with the largest: the blue cube of millions) and showing similar figures in groups.

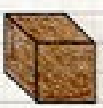
2. Describe the hierarchical value of each figure, writing with the child:

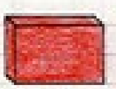


100^3 = 100 X 100 X 100 = h X h X h = M = 1,000,000

(3)  $3(100^2$ X 100) = 100 X 100 X 10 = h X h X t = Hk = 300,000

(3)  $3(100^2$ X 1) = 100 X 100 X 1 = h X h X u = Tk = 30,000

(3)  $3(10^2$ X 100) = 10 X 10 X 100 = t X t X h = Tk = 30,000

(6)  6(100 X 10 X 1) = 100 X 10 X 1 = h X t X u = k = 6,000

 10^3 = 10 X 10 X 10 = t X t X t = k = 1,000

(3)  $3(1^2$ X 100) = 1 X 1 X 100 = u X u X h = h = 300

(3)  $3(10^2$ X 1) = 10 X 10 X 1 = t X t X u = h = 300

(3)  $3(1^2$ X 10) = 1 X 1 X 10 = u X u X t = t = 30

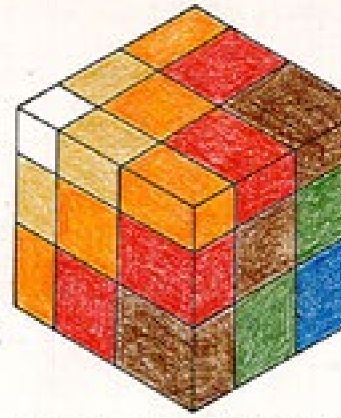
 1^3 = 1 X 1 X 1 = u X u X u = u = 1

The Hierarchical Sum: 1,367,631

To further indicate the hierarchical sum, we group the figures according to color, noting those hierarchies which are represented by two different figures, but whose hierarchy is indicated by like colors.

Cube Root. . .

N. Raising the Trinomial to the Third Power



The sensorial construction is through the diagonal, constructing first the three dimensions (two sides and the altitude) AND completing each successive hierarchy. In this way, we have a color control for the construction.

We begin with the blue cube (millions). Then the three green prisms are added on three sides of that cube, and EVERYTHING IS GREEN. Next we go to the brown prisms (ten thousands), adding the (second digit) dimension to the three dimensions. THEN WE MUST COMPLETE THE TEN THOUSANDS HIERARCHY, so the second group of three brown prisms is added (in the first confirmation position) and EVERYTHING IS COVERED BROWN.) Next the six prisms of red thousands are added and the red cube (of 10^3) and EVERYTHING IS RED. The addition of the first group of orange prisms (hundreds) completes the dimensions of the two sides and altitude (third digit position) and then when the next group of orange prisms is added, we turn EVERYTHING TO ORANGE. The tan prisms, tens, give us the second confirmation of the third digit and the last placement (third confirmation) is that of the units hierarchy, the white cube.

The Hierarchical Cube of the Trinomial
$(h + t + u)^3$

$$
\begin{array}{llcr}
 & (642)^3 & & \\
= & (600 + 40 + 2)(600 + 40 + 2)(600 + 40 + 2) & & \\
= & 600^3 & = & 216,000,000 \\
+ & 3(600^2 \times 40) & = & 43,200,000 \\
+ & 3(600^2 \times 2) & = & 2,160,000 \\
+ & 3(40^2 \times 600) & = & 2,880,000 \\
+ & 6(600 \times 40 \times 2) & = & 288,000 \\
+ & 40^3 & = & 64,000 \\
+ & 3(2^2 \times 600) & = & 7,200 \\
+ & 3(40^2 \times 2) & = & 9,600 \\
+ & 3(2^2 \times 40) & = & 480 \\
+ & 2^3 & = & 8 \\
\hline
 & & & 264,609,288
\end{array}
$$

In this work, the construction of the cube and the calculation are done simultaneously.

O. The Algebraic Cube of the Trinomial

Material: The algebraic cube of the trinomial. Here there is no hierarchical color guide. The colors are, rather, a reference to a, b, and c. And only those faces of the figures which show the square of the particular letter are colored. Other faces, representing the product of two different terms, are black. The colors, however, continue to act as an aid in construction because color faces must be matched.

1. Display the 27 figures of the algebraic cube for the trinomial:

$a^3 \quad a^2b \; a^2b \; a^2b \quad a^2c \; a^2c \; a^2c \quad abc \; abc \; abc \; abc \; abc \; abc \quad b^3 \quad ab^2 \; ab^2 \; ab^2 \longrightarrow$

$\longrightarrow b^2c \; b^2c \; b^2c \quad ac^2 \; ac^2 \; ac^2 \quad bc^2 \; bc^2 \; bc^2 \quad c^3$

2. Bring forward the three cubes and state the problem, naming the cubes. $(a + b + c)^3$

2. Now instead of giving the trinomial cube a numerical value, we will give it letter values.

3. Using small cards for the symbols and signs, lay out the problem on the mat:

$$(a + b + c)^3 = (a^2 + \frac{ab}{ab} + \frac{ac}{ac} + b^2 + \frac{bc}{bc} + c^2) \times (a \times b \times c)$$

We must multiply all the terms within the parenthese times a, times b and times c.

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A GRATEFUL ACKNOWLEDGEMENT to Phil Gang for the original drawings of the "Passages to the Cube Root." Bergamo. 1974

Cube Root. . .
O. The Algebraic Trinomial. . .

4. The construction of the algebraic trinomial raised to the third power is done "layer by layer," as indicated in the figure shown.

Each layer is built on a base "guide card," as in the construction of the algebraic binomial raised to the third power. Now, however, the guide card corresponds to the square of 111; again the card acts as a guide in the placing of the figures.

$\times a$

$a^2 \times a = a^3$	$ab \times a = a^2b$	$ac \times a = a^2c$
$ab \times a = a^2b$	$b^2 \times a = ab^2$	$bc \times a = abc$
$ac \times a = a^2c$	$bc \times a = abc$	$c^2 \times a = ac^2$

When all terms are thus multiplied by a, it is returned face down to the equation, and a second guide card begun for "b."

5. Dissemble the cube, lining the figures again horizontally on the mat, as in the first display: SHOW THE CORRESPONDING LETTER CARD ON EACH FIGURE.

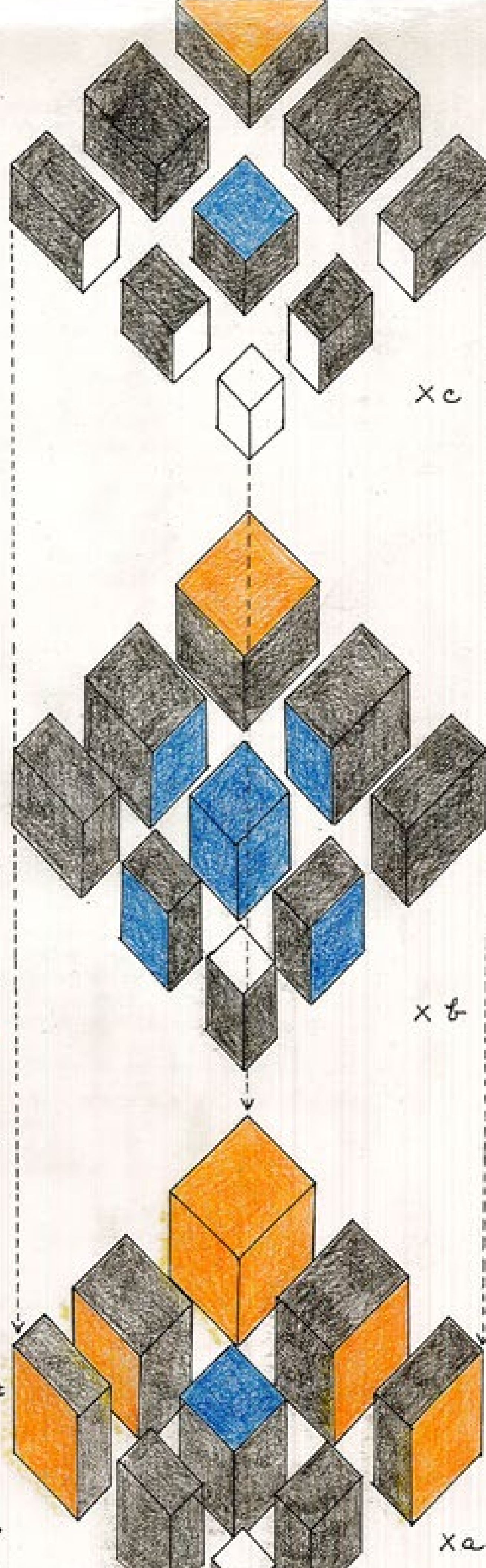
6. Then combine the cards in front of the figures:

$$a^3 + \begin{matrix}a^2b\\a^2b\\a^2b\end{matrix} + \begin{matrix}a^2c\\a^2c\\a^2c\end{matrix} + \begin{matrix}abc\\abc\\abc\\abc\\abc\\abc\end{matrix} + \dots\dots$$

7. Introduce new cards which simplify the equation and SHOW THE FIGURES GROUPED ACCORDING TO THE EXPRESSION:

$$a^3 + 3(a^2b) + 3(a^2c) + 6(a \times b \times c) + b^3$$
$$+ 3(b^2a) + 3(b^2c) + 3(c^2a) + 3(c^2b) + c^3$$
$$= (a + b + c)^3$$

8. Formulate the rule: The cube of the trinomial is equal to the cube of the first term plus three times the product of the square of the first term times the second term plus three times the product of the square of the first term times the third term plus six times the product of the three terms plus. . . . .



8) Confirmation of third digit. Cube construction indicates three groups of the 2^2 taken seven times. (FAT ORANGE PRISMS OF HUNDREDS: $3(ac^2)$)

9) Confirmation of the third digit. Cube construction indicates three groups of 2^2 taken three times. (NEW DIGIT BROUGHT DOWN: YELLOW PRISMS: $3(bc^2)$)

10) BRING DOWN FINAL DIGIT. Final confirmation. Cube of 2. (WHITE CUBE: c^3)

NOTE: We work with only a few examples of this construction due to the difficulty of handling the tiny neutral cubes.

NOTE: Throughout this construction, we have given hierarchical value to the wooden materials which originally represented only units. We are thus equipped to pass easily onto the work with the hierarchical trinomial.

Presentation #7: **The Cube of the Trinomial with the Hierarchical Cube**

$\sqrt[3]{160\ 103\ 007}$	h t u 5 4 3	
125	$5^3 = 125$	First digit
351	$5^2 \times 3 = 75$; $351 \div 75 = 4$	
300	$75 \times 4 = 300$	Second digit
510	$3(4^2 \times 5) = 240$	
240		
270	$5^2 \times 3 = 75$; $270 \div 75 = 3$	
225	$75 \times 3 = 225$	Third digit
453	$6(5 \times 4 \times 3) = 360$	
360		
93	$4^3 = 64$	
64		
290	$3(4^2 \times 3) = 144$	
144		
146	$3(3^2 \times 5) = 135$	
135		
110	$3(3^2 \times 4) = 108$	
108		
27	$3^3 = 27$	
27		
0		

In this work, we continue to bring down one digit each time the hierarchy changes, or is completed. And this completion is indicated by the material itself. The material continues to be the guide for the order of construction. And finally, it gives the hierarchical value to the work which is still lacking in the calculation itself.

Presentation #8: **Moving Towards Abstraction: The Trinomial Cube Root with the Algebraic Cube Materials**

$\sqrt[3]{80\ 621\ 568}$	h t u 4 3 2		
64	$4^3 = 64$	First digit	a^3
16621	$4^2 \times 3 = 48$; $166 \div 48 = 3$		
	$3(400^2 \times 30) + 3(30^2 \times 400) + 30^3 =$		$3(a^2b)+3(ab^2)+b^3$
	$14,400,000 + 1,080,000 + 27,000$		
$-15,507,000$	$= 15,507,000$	Research & confirmation of 2nd digit	
1 114 568	$43^2 = 1849 \times 3 = 5547$		
	$11145 \div 5547 = 2$	Research of third digit	
	$3(400^2 \times 2) = 960,000$	Confirmation of third digit.	$+3(a^2c)$
	$+6(400 \times 30 \times 2) = 144,000$		$+6(abc)$
	$+3(2^2 \times 400) = 4,800$		$+3(c^2a)$
	$+3(30^2 \times 2) = 5,400$		$+3(b^2c)$
	$+3(2^2 \times 30) = 360$		$+3(c^2b)$
	$+2^3 = 8$		$+c^3$
$-1,114,568$	$1,114,568$		

Cube Root. . .
The Trinomial Cube Root with the Algebraic Material. . .

In this passage, we move to abstraction. The groups of three digits are brought down at one time. In the research of the second and third digits, however, the last two digits are separated to indicate that the multiplication and division operations done to research those digits are not hierarchical ones. This is an indication, too, of the actual volume which is available to determine the dimensions of the cube; the last two digits act as confirmation digits.

The use of the algebraic cube is important in this operation because it gives a guide for the calculation of the confirmations. No longer will the hierarchical materials be adequate for the calculation that includes the entire confirmation. The pattern for that confirmation is given in the algebraic representation.

We note that with the confirmation of the second digit, our resulting construction is a whole cube, the root of which is 43. (Hierarchically this is 430, but the digits are indicated without hierarchical value. Thus, in our search for the third digit, we have $43^2 \times 3$: we indicate that we must build in three dimensions on the side of the whole first cube.

Presentation #9: To Complete Abstraction: Without materials. . .although the children may continue to use either the hierarchical materials or the algebraic cube as a reference and visual guide.

	h t u	
$\sqrt[3]{80\ 621\ 568}$	4 3 2	
64	$4^3 = 64$	First digit
166	$4^2 \times 3 = 48$	
	$166 \div 48 = 3$	Second digit
80 621	$43^3 = 79507$	
− 79507		Second digit confirmation
11145	$43^2 = 1849$; $1849 \times 3 = 5547$	
	$11{,}145 \div 5547 = 2$	Third digit
80 621 568		
− 80 621 568	$432^3 = 80{,}621{,}568$	Third digit confirmation
0		

PARTICULAR CASES OF THE CUBE ROOT

Introduction: As a preparation for the particular cases of the cube root, we review with the children the work of "squaring" and "cubing" with the materials (peg board/hierarchical pegs and hierarchical cube), the child writing the corresponding notation:

$10^2 = 10 \times 10 = 100$ (RED PEG)

$10^3 = 10 \times 10 \times 10 = 1{,}000$ (RED CUBE of the hierarchical binomial cube.)

****

$$11^2 = (10 + 1)^2 = 10^2 + 2(10 \times 1) + 1^2 = 100 + 20 + 1 = 121$$

(PEG CONSTRUCTION: GUIDE CHART comparison of 11^2)

$$11^3 = (10 + 1)^3 = 10^3 + 3(10^2 \times 1) + 3(1^2 \times 10) + 1^3 = 1{,}000 + 300 + 30 + 1 = 1{,}331$$

(HIERARCHICAL BINOMIAL CUBE)

****

$100^2 = (100 \times 100) = 10{,}000$ (BLUE PEG)

$100^3 = (100 \times 100 \times 100) = 1{,}000{,}000$ (BLUE CUBE OF HIERARCHICAL TRINOMIAL)

Cube Root. . .
Particular Cases: Introduction (Review). . .

$$111^2 = (100 + 10 + 1)^2$$
$$= 100^2 + 2(100 \times 10) + 2(100 \times 1) + 10^2 + 2(10 \times 1) + 1^2$$
$$= 10{,}000 + 2{,}000 + 200 + 100 + 20 + 1$$
$$= 12{,}321$$

(PEG SQUARE)

$$111^3 = (100 + 10 + 1)^3$$
$$= 100^3 + 3(100^2 \times 10) + 3(100^2 \times 1) + 3(10^2 \times 100) + 6(100 + 10 \times 1)$$
$$+ 10^3 + 3(1^2 \times 100) + 3(10^2 \times 1) + 3(1^2 \times 10) + 1^3$$

$$= 1{,}000{,}000 + 300{,}000 + 30{,}000 + 30{,}000 + 6{,}000 + 1{,}000 + 300 + 300 + 30 + 1$$
$$= 1{,}367{,}631$$

(TRINOMIAL CUBE)

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Particular Case: Constructing the Square and the Cube of 101: Preparation

$$101^2 = (100 + 10 + 1)^2 = 100^2 + 2(100 \times 0) + 2(100 \times 1) + 0^2 + 2(0 \times 1) + 1^2$$
$$= 10{,}000 + 0 + 200 + 0 + 0 + 1$$
$$= 10{,}201$$

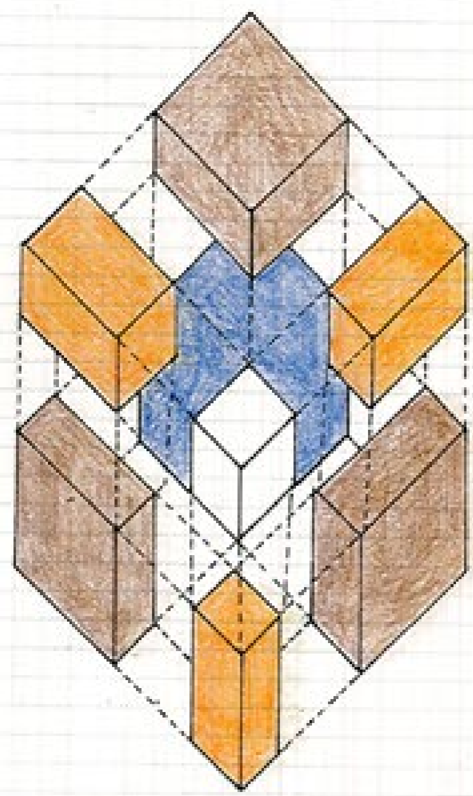
(PEG CONSTRUCTION OF THE SQUARE AS SHOWN IN PASSAGES TO THE SQUARE ROOT)

INTRODUCE THE TRANSPARENT TRINOMIAL CUBE: With these materials we can show, in our special construction of the trinomial cube, those hierarchies which are zero and therefore are not present even though they exist. DISPLAY THE FIGURES OF THE HIERARCHICAL TRINOMIAL CUBE, grouped according to hierarchies in a horizontal row. And, in front of each group, show that corresponding transparent figure which will be substituted whenever it is shown in the calculation to be zero:

$$101^3 = (100 + 0 + 1)^3$$

$= 100^3$	(Blue cube)
$+ 3(100^2 \times 0)$	(Transparents)
$+ 3(100^2 \times 1)$	(Tall Brown)
$+ 3(0^2 \times 100)$	(Transparents)
$+ 6(100 \times 0 \times 1)$	(Transparents)
$+ 0^3$	(Transparent)
$+ 3(0^2 \times 1)$	(Transparents)
$+ 3(1^2 \times 100)$	(Tall orange)
$+ 3(1^2 \times 0)$	(Transparents)
$+ 1^3$	(White cube)

$$= 1{,}000{,}000 + 0 + 30{,}000 + 0$$
$$+ 0 + 0 + 0 + 300 + 0 + 1$$
$$= 1{,}030{,}301$$



When the construction is completed, we can take out the entire middle because the tens digit was zero. Slide out the two center columns, then finally take out the entire second layer. What remains is: First layer: blue cube, tall brown prisms, and orange prism; and Second layer: Brown prism, two orange prism and white cube. Representing: $a^3 + 3(a^2c) + 3(c^2a) + c^3$. And that gives us our calculated sum.

Cube Root. . .
Particular Cases. . . Introduction: Review. . .

Particular Case #2: Constructing the Square and the Cube of 110: Preparation

$$110^2 = (100 + 10 + 0)^2 = 100^2 + 2(100 \times 10) + 2(100 \times 0) + 10^2 + 2(10 \times 0) + 0^2$$
$$= 10{,}000 + 2{,}000 + 0 + 100 + 0 + 0$$
$$= 12{,}100$$

(PEG CONSTRUCTION OF THE SQUARE: SEE SPECIAL CASES OF THE SQUARE ROOT)

$$110^3 = (100 + 10 + 0)^3$$

$= 100^3$	(Blue cube)
$+ 3(100^2 \times 10)$	(Green prisms)
$+ 3(100^2 \times 0)$	(Transparents)
$+ 3(10^2 \times 100)$	(Fat brown prisms)
$+ 6(100 \times 10 \times 0)$	(Transparents)
$+ 10^3$	(Red cube)
$+ 3(10^2 \times 0)$	(Transparents)
$+ 3(0^2 \times 100)$	(Transparents)
$+ 3(0^2 \times 10)$	(Transparents)
$+ 0^3$	(Transparent)

$$= 1{,}000{,}000 + 300{,}000 + 0 + 30{,}000 + 0 + 1{,}000 + 0 + 0 + 0 + 0$$
$$= 1{,}331{,}000$$

Because the units digit is zero, the construction of the cube results in a cube of which the entire external part is transparent; that is, the two terminal sides of the construction and the top layer. We can then remove those sides and top and what remains is the cube of the binomial construction. . .but in this case we see a different hierarchical pattern; and we note that in reading the cube root from this construction it is necessary to add a zero which is the final digit represented by the transparent part.

Particular Cases: REAL CUBE ROOT

In the cube root work with the particular cases, we again utilize the transparent cube along with the corresponding hierarchical cube. The transparent figures are inserted into the construction whenever the result of our operation, as shown in the calculation, is zero.

Preparation:

	h t u
$\sqrt[3]{1\ 367\ 631}$	1 1 1
1	$1^3 = 1$
0 3'67	$1^2 \times 3 = 3;\ 3 \div 3 = 1$
331 000	$3(100^2 \times 10) + 3(10^2 \times 100) + 10^3 = 331{,}000$
36 6'31	$11^2 \times 3 = 363;\ 366 \div 363 = 1$
36631	$3(100^2 \times 1) + 6(100 \times 10 \times 1) + 3(1^2 \times 100) + 3(10^2 \times 1)$
	$+ 3(1^2 \times 10) + 1^3$
	$= 30{,}000 + 6{,}000 + 300 + 300 + 30 + 1$
	$= 36{,}631$

Particular Case #1

	h t u
$\sqrt[3]{1030301}$	1 0 1
1	$1^3 = 1$
0 0'30	$1^2 \times 3 = 3;\ 0 \div 3 = 0$
0	$3(100^2 \times 0) + 3(0^2 \times 100) + 0^3 = 0$
30 3'01	$10^2 \times 3 = 300;\ 303 \div 300 = 1$
30301	$3(100^2 \times 1) + 6(100 \times 0 \times 1) + 3(0^2 \times 1) + 3(1^2 \times 100)$
	$+ 3(1^2 \times 0) + 1^3$
	$= 30301$