Second Level: In two parts: Without the Circle and With the Circle: AGE: 9---
PROBLEMS: Putting Into Relationship the Calculation of the Area of Different Figures.

Material: The series of squares, circles and the corresponding curvalinear square.

A strange man wants to arrange circular bottles in a cupboard. We know the dimensions of the bottles and the dimensions of the cupboard. That is, the circles and the squares available. How will I know how many circles will fit in the cupboard?

Solution: To determine the number of bottles, we must divide the area of the shelf by the area of the bottle.

$A_{cupboard} \div A_o$ = Number of bottles We have GROUP DIVISION.

If the shelf dimensions = 60 X 40
then $A_{cupboard}$ = 2,400
$A_o = \pi r^2$
$r = 5$
$A_o = \pi\ 25 = 78.5$
So $2400 \div 78.5 = 31$ r. 66.5

BUT, laying out those circles on the pavement (our board) which is 60 X 40, we see that we can only fit 6 X 4, or 24 circles.

Why?

Because I cannot divide the total area of the space by the area of only the circle. I must also use that area which circumscirbes each circle: our curvalinear square. SUPERIMPOSING the circle on the square we can see FOUR EMPTY SPACES, the sum of which is that curvalinear square. I can put four circles together to note the internal figure which is equal to one of those empty spaces (one of the corners) X 4.

So: $A_{cupboard} = A_{square} - A_o$

Third Level: To calculate the area of the other figures necessary to cover the surface, we must go on to trigonometry. . . . . later.

VOLUME

Material
1. The blue (red) number rods.
2. The broad (brown) stair.
3. The pink tower.
4. The solid cylinders (solid insets).
5. Box of small geometrical solids.
6. A wooden box of 250 netral cubes: each 2 cm.^{3}
7. The rectangular solid and 5 rectangular solids whose sum is equal to the first square-based rectangular solid: yellow and having similar markings to the yellow area material. (SEE FIGURE) In a large wooden box.
8. The cubes from the cabinet of powers.
9. Ten hundred squares.

INTRODUCTION

In the study of volume, we follow the same pattern for the study of areas because the starting point is the same.

Has the child had the concept of volume before? We divide the study into two parts: the concept of solids and the actual measuring of the solids which is volume.

The concept of the solid as a body occupying space is one the child has in some activities in children's house. The education of the visual sense includes a perfection of the concept of size. This sense has been developed with: 1) the blocks; that is, the rods, the stair and the cubes (of the pink tower); and 2) the solid inset series, the cylinders in which we repeat the same characteristics of the rod, the stair and the cube. In casa he also has experience in the concept of solids with the small geometrical solids; training the stereognostic sense as well as the visual experience of size. He has also had experiences with solids in the world itself, a consciousness that everything in the world occupies space.

Volume. . .
Introduction. . .

The concept of volume, the measuring of these solids know at home and school, has been introduced through experiences with the decimal system. In the first presentation of the decimal system materials we present the cube: "This is a thousand." And then in the second presentation we stack ten hundred squares to create the thousand. The child has started constructing the solid as soon as he adds one square to the next. And so the child has received helps from nature and from the method.

The only thing we must bring to the child's consciousness is that everything in our world is composed as a solid, occuping space and having three dimensions. This is the measure of reality which our stereognostic sense records.

Presentation #1: The Solid and its Transformation at the Level of Area and Equivalence

Material: The box of 250 (2 cm^3) cubes.

1. Take twelve cubes from the box and FIRST CONSTRUCT A ROD.
Note that with this formation of the cubes, we can show only two positions.
Vertical and horizontal: 1 X 1 X 12.

1. Let's try to construct something with these cubes.
We see that every face is a square.
This are the only positions this form can take.

2. Form the parallelopiped: 3 X 4 X 1.
Show in three positions.

2. We can unite these twelve cubes in this way: now we have a figure half the length of the first and double the width.
And we can show this form in THREE POSITIONS.

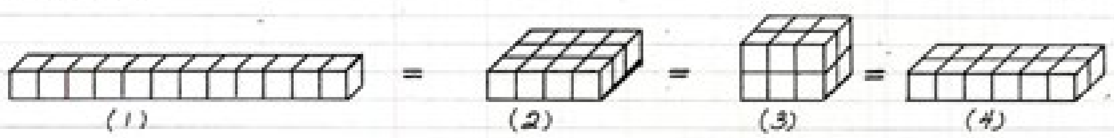
3. Construct the rectangular prism:
2 X 2 X 3

3. This form has only two possible positions.

4. Construct the parallelopiped: 1 X 2 X 6: again we can show three positions.

5. CONCLUSIONS: The more divisors the form has, the more positions we can show; the more variables the work has.
ALL THE FIGURES ARE FORMED OF 12 CUBES: therefore, they all have EQUIVALENT VOLUME.

NOTE: Use a polaroid to show all the different forms simultaneously, emphasizing the equivalence.



Presentation #2: Volume of the Parallelopiped

Material
1. The wooden box containing: the yellow square-based rectangular solid:
10 X 10 X 20 cm^3
and : five yellow rectangular parallelopipeds, whose sum is equal to the first rectangular solid.

1. Take from the broad stair the thickest stair and classify: a prism.

2. Compare the yellow prism (of the new material) with the stair, verifying equivalence. Put the brown one aside: the yellow now is the mediator.

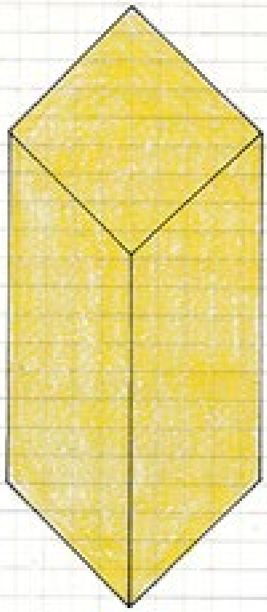
3. THE PROBLEM: How can I measure this prism?

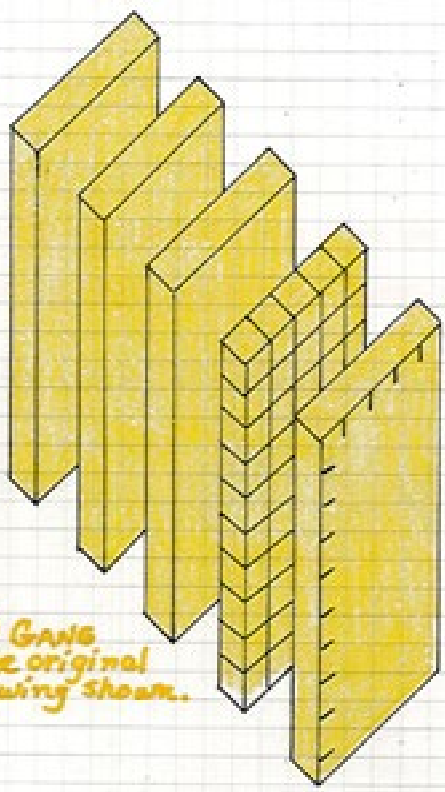
3. This is a square. (Looking at the top)
We cut this square into slices, using the same unit of measure we used for the areas.
We would mark off 2 cm. line segments, and cut vertically with a knife.

Introduce the five rectangular prisms:

We obtain five slices: the height is the same; taken together the base is the same.
I cannot measure the volume with these slices unless my figure is made specifically of slices. . .so

3. . .the problem of calculating volume.





THANKS to Phil GANG for the original drawing shown.

3. . So I transfer my unit of measure to two sides, resulting in these prisms divided horizontally and vertically, into long and short rectangles.
I obtain this prism, divided into cubes.
Each edge of each cube is equal to the unit of measure we have used to make every cut.
THE CUBE is out unit of measure for volume.

(VOLUME NOMENCLATURE:
- ovoid
- ellipsoid
- sphere
- right circular cone
- equilateral cylinder
- regular hexagonal right prism
- right pentagonal prism
- cube (regular hexahedron)
- regular triangular pyramid
- right rhombic parallelopiped
- rectangular parallelopiped (rectangular solid
- square-based rectangular solid
- right circular cylinder
- right square pyramid

Casa nomenclature:
- ovoid
- ellipsoid
- sphere
- cone
- cylinder
- square pyramid
- triangular pyramid
- cube
- square prism
- triangular prism)

4. Show the box containing the 250 cubes:

4. If I want to calculate the volume of this rectangular solid, how many of these cubes are contained in that solid?
We could remove the cubes here and in their place fit the rectangular solid.
We have divided the solid into units (cubes) that can be counted.

5. Consider the number of cubes contained in one of the five rectangular parallelopipeds created by the slicing of the square-based rectangular solid: use that one divided into cubes. Then indicate the multiplication for total volume.

Show the single neutral cube as the unit.

5. We can count the number of cubes in one of these five sections: 50.
There are 50 cubes here as there were 50 squares in the rectangle in area.
But in this square-based rectangular there are five sections. . .so we can count the cubes: 50, 100, 150, 200, 250. 250 cubes that all look like this.

Volume of the Parallelopiped. . .

NOTE: The pedagogist F. Frobel created didactic materials which provided an indirect preparation for some of the work of Dott.sa Montessori. He spoke of his own work as his gifts. And among those gifts were what he called "The Prince Solids:" the sphere and the cube and the cylinder.

The sphere is limited by a curved surface; the cube by planes. So the cylinder provides us with a mediator between the cube and the sphere (plane surfaces and curves). The cylinder is composed of lateral surfaces which are curves and bases which are planes.

Dott.sa Montessori takes Frobel's cube which is divided into bricks. (SEE The Discovery of the Child). Montessori on Frobel: The cube and the sphere are called the constructors of reality because all things are limited by either the plane surface or that curved one of the sphere. Frobel's obsession with these three figures is total.

And Dott.sa Montessori, separating herself from some of the older concepts, keeps certain parts of his work: the concept that the sphere and the cube are perfect solids. The perfection of the sphere is primitive: all planets are spheroid. The perfection of the cube is given by the fact that in any position it is the same. And all cubes and spheres of reality are similar.

At the level of area all triangles, squares and regular polygons are similar among themselves. In area, then, we take the perfect square as the measure. Given the perfection of the cube, we have taken it as the measure of space, of volume. We can take ANY CUBE as the unit of measure in the same way that we give the specific square which is our measure of area no specific dimension. Any square was sufficient; any cube in volume will serve.

The point of consciousness here, then is that we must decompose the solid into cubes and then the number of cubes (the sum of those cubes) is the measure of the volume of that solid. The important thing is that we count the cubes; and their sum is the volume.

Presentation #3: Arriving at the Formula for the Volume of the Parallelopiped

1. Show the square-based rectangular solid and the five sections: STATE THE PROBLEM: How can we arrive at the sum of 250 in a simpler way? SHOW THE BOX OF CUBES, indicating that we could count them all.

1. We know that in each of these sections there are 50 cubes; and therefore, this square-based rectangular solid is composed of 250 cubes. But we need a faster method of discovering this figure than counting all the cubes.

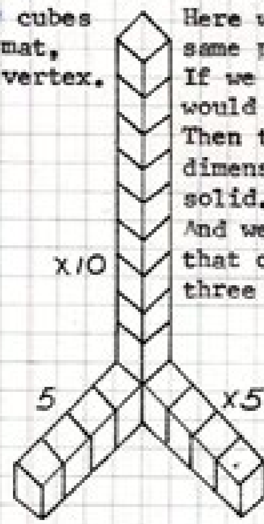
2. Recall the solution at the level of area.

2. In out work with area, we discovered that we could obtain 50 by multiplying one side of 5 by the other of 10: 5 X 10 = 50.

3. Show the solution of the product of the three dimensions: Take 9 cubes and form a right angle on the mat, then add 9 more on the common vertex.

3. How are these 250 cubes obtained? Here we have three edges concurrent to the same point: the common vertex. If we filled the figure in, how many cubes would we have? 250 Then these three edges represent the three dimensions of our square-based rectangular solid. And we have calculated the sum of the cubes that compose this solid by multiplying the three edges times each other:

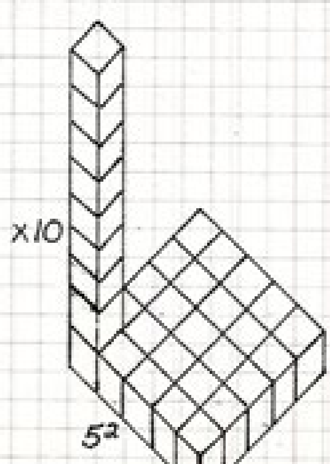
5 X 5 X 10 = 250



4. Second solution: construct a square, side of 5 cubes, then show nine more cubes on on vertex:

$5^2 = 25 \times 10 = 250$

5. Third solution: Show the five sliced sections: these represent a multiplication of: 50 X 5 = 250



6. CONCLUSIONS: We have organized three ways to calculate the volume of this solid:

1) 250 = 5 X 5 X 10 — The product contains the three dimensions of reality.

2) 250 = 5^2 X 10 = (5 X 5) X 10 — Here we see those three elements grouped in a different way.

3) 250 = (10 X 5) X 5 — Again the elements are grouped differently, but all three are present in the calculation.

For the calculation of volume we will use only the first and second solutions, not the third. We do not use the third because it requires a certain orientation of the square-based rectangular solid, the vertical one. (Here we have used that position because the two bases are equal and we take the name from these two faces; with the sliced sections of the figure, then, we construct the vertically-positioned solid.) Then, the elements of the solution (10 X 5 X 5) give the dimension of the height first, and this must be the LAST element of our formula.

The first and second products provide a formula in which the three dimensions are named in correct order: a X b X c . . .where a is one dimension of the base, b is the second dimension and c is the height:

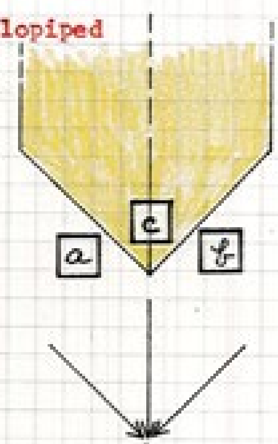
1) 5 X 5 X 10 = a X b X c = base X base X height

2) (5 X 5) X 10 = (a X b) X c = product of the area of the base X height

We see that our formula must include the elements in a certain order: the area of the base must be given first times the third dimension.

SO. . .we conclude: we have discovered two ways to calculate the volume of this solid: by multiplying the three dimensions (a X b X c) AND by multiplying the area of the base times the height (a X b) X c.

Presentation #4: The Formula for the Volume of the Parallelopiped



1. Showing the yellow square-based rectangular solid, give the nomenclature.

2. Introduce the letters shown on the figure:
   Everything that forms this solid is given by the product of a, b, and c---we call this product of the three elements VOLUME.

3. Show on the mat the formula: V = a b c
   In the formula we have indicated the dimensions following the EDGES to the COMMON VERTEX: we can touch it three times: no more.

4. Rearrange the formula with parentheses:

   V = (a b) X c

   The volume of the solid is also given by the area of the base times its height. So we can show the formula in a different way with parentheses.

5. Invite the child to take a unit of measure, that one used for the measurement of the area of our first rectangle in the yellow material of area (2 Cm.); and to measure a, b, and c. As he measures each of the dimensions of the solid, he takes the corresponding slip and writes on the back the number of units:

   a = 5
   b = 5 — Now the formula reads: V = (5 X 5) X 10
   c = 10 — Removing parentheses: V = 5 X 5 X 10

   And we note again that the third dimension is always the third term: height.

6. Introduce the correct nomenclature and its significance in volume work: *that is, we will learn to calculate the volume of other solids in terms of this one.*

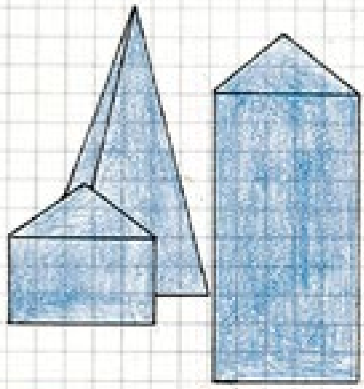
6. This square-based rectangular solid is called the square-based right-angled parallopiped.
   This is our point of reference at the level of volumes just as the rectangle was for area.

Materials for the Study of the Volume of other Solids

1. A Box #1 containing:
   a) Two regular triangular right prisms; one whole and one divided into two smaller regular right triangular prisms. . by tracing an altitude of the first from one vertex to the opposite base.
   b) A right rhombic parallelopiped.
   c) Two regular hexagonal right prisms, one whole and the other divided into three parts by the joining of two non-adjacent vertices and then by tracing an altitude of the resulting isosceles triangle. (SEE presentation)

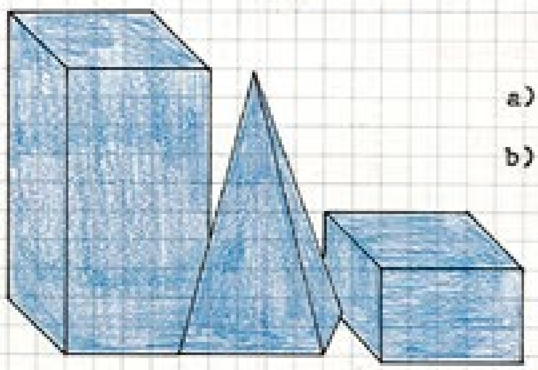
2. A Box #2 containing:

Family #1:



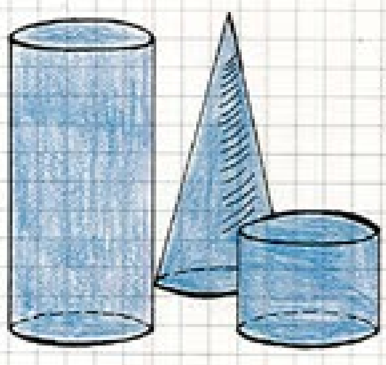
a) right triangular prism
b) right triangular prism, h = 1/3
c) regular triangular pyramid

Family #2:



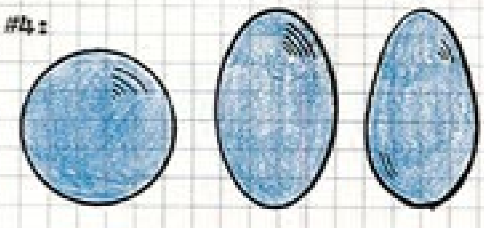
a) square-based rectangular solid (right parallelopiped)
b) square-based rectangular solid, h = 1/3
c) right square pyramid

Family #3:



a) right circular cylinder
b) right circular cylinder, h = 1/3
c) right circular cone

Family #4:



a) sphere
b) ellipsoid
c) ovoid

3. A Box #3 containing: HOLLOW FIGURES OF:
   a) tall rectangular parallelopiped
   b) short rectangular parallelopiped
   c) right square pyramid
4. Wooden box containing the cube divided into THREE EQUAL TRIANGLES.

INTRODUCTION: To the Calculation of the Volume of Other Solids

We know that the solids can be divided into three groups: prisms, pyramids and solids of rotation (rotated bodies.) We begin our work of volume with the prisms taken from the Box #1.

Presentation #1: The Volume of the Right Triangular Prism

Materials

1. The yellow rectangular parallelopiped.
2. From box #1, the blue regular triangular right prisms (whole and that one divided into two prisms.)

1. Present both the rectangular parallelopiped and the right triangular prism: Name both as PRISMS. Show both figures horizontally resting on each of the lateral surfaces. Then orient vertically.

   1. These are both prisms. First we must orient our solids. We can show them horizontally, the triangular prism in three positions that each look the same. The lateral surfaces of both prisms are rectangles.

2. Show both prisms on their bases, giving complete nomenclature for the triangular prism.

   2. We must orient the solids on one of the equal bases: these two faces are <u>opposite</u> each other which determines them as the bases. Because of its position, we give it a name:

This is a regular. . .because the base is an equilateral triangle
right-angled . . .because its edges form a right angle with the plane
triangular prism.

3. Compare the heights of the two prisms. NOTE: We note here that the third dimension is secondary. The real secret lies, then, in the AREA study of the regular polygons which gives us the calculation for the bases.

   3. These solids have the same height, but because their names differ, we know that their bases will differ. The secret of the solids lies in the base.

4. The PROBLEM: how do we calculate the volume of the regular right-angled triangular prism?

   4. We have understood that we cannot divide this figure exactly into cubes. . .a problem similar to the division of the triangle into squares in our area work.

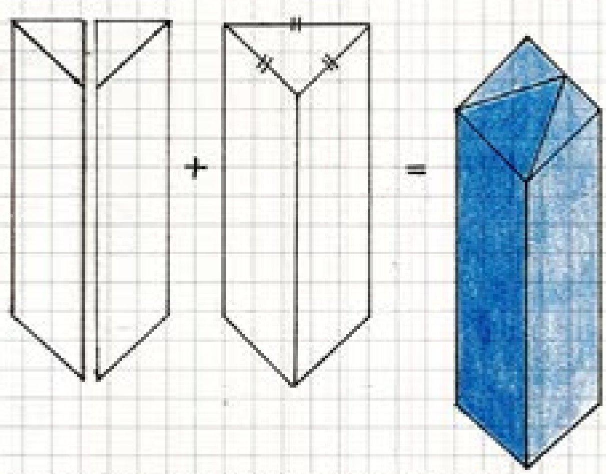
5. Introduce the MEDIATOR: the right triangular prism which has been divided into two right triangular prisms by a line traced from the altitude of one vertex to the opposite base and then a cut. SHOW CONGRUENCE. . .and then combine the two to compose the rectangular parallelopiped.

   We have two equal solids; we want to calculate the vol. of only one.

   We begin by doubling that one figure:

   Can we decompose this into cubes?

   Y E S.



6. Introduce again the yellow prism. Compare the newly constructed one with that prism, noting that WE DO NOT HAVE THE SAME FIGURE.

   NOTE: If our starting point is the base of the equilateral triangle, then we know that the height is not equal to the side: $h = \frac{s\sqrt{3}}{2}$

   The height is, in fact, always shorter than the side: $a < s$

   In the isosceles triangle, we can construct h = s. . .

   a triangle, the double of which would give us our square-based rectangular solid. That is, we needed h = s. But here we must work with the equilateral base, so we have a NEW (s) side.

6..NOTE. . .with the children that when we calculate volume, it is important to have a figure whose bases are formed of edges perpendicular to one another. That is, it does not necessarily have to be a square, but simply right angles on the base so that we can decompose the figure into cubes. So. . .this new rectangular base will serve.

7. Compose the formula:

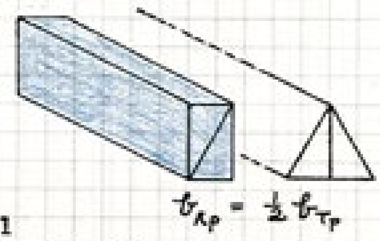
$$V_{Tp} = \frac{A_b \times h}{2}$$

$$V_{TP} = \frac{A_b \times h}{2}$$

7. From this figure we can compose the formula for the volume.
We write the area of the base X the height.
BUT OUR FORMULA IS INCOMPLETE, because it represents the volume of TWO figures.
So. . .for the volume of one regular right triangular prism, we must divide that produce by 2.

8. REMOVE THE MEDIATOR: now we work with ONLY THAT TRIANGULAR PRISM composed of the two pieces. We have a new problem: to form a solid with its base angles perpendicular using only these two pieces; that is, ONE RIGHT TRIANGULAR PRISM.
   a) Using these two pieces (prisms) we first discover a prism with another TRIANGLE as the base. And we discover one with a DELTOID as base.
   b) As in the constructive triangles (Series #1, Box #2), we discover that the secret is turning over one piece: now we may discover TWO PARALLELOGRAMS as the base of the prism.
   c) And finally we arrive at the solution: a rectangular parallelopiped with base angles that are perpendicular. . .

   . . .this rectangular parallelopiped is equivalent to the regular right triangular prism.
   So. . . $V_{Tp} = A_b \times h$

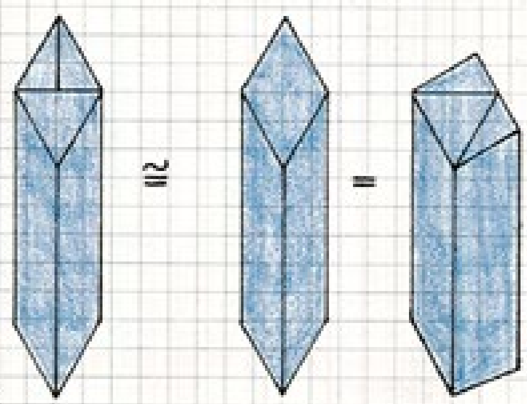


   d) Now we must identify b. . .in terms of the original figure. What is our base? Turn over the piece and again form the triangular prism. Now we can show that we have divided the base of the triangular prism into two equal parts. So the base of the newly formed rectangular parallelopiped has $b = \frac{1}{2}b_{tp}$. And the formula for the AREA of the BASE of the parallelopiped will be: $A = \frac{b}{2} \times h$
   e) So we recompose the formula: $V_{Tp} = (\frac{b}{2} \times h) \times h$ OR $A_b \times h$

Presentation #2: The Volume of the Right Rhombic Parallelopiped

1. Introduce the figure. Give the name. Note again that the figure takes its name from the opposing faces, or bases.

1. This is a right rhombic parallelopiped. Can we decompose this solid into cubes? No.



2. Introduce the MEDIATORS: the two right triangular prisms. Put the two together and VERIFY CONGRUENCE, showing the equality of the bases and heights between the two figures.

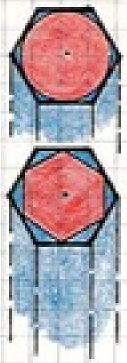
3. Put the rhombic parallelopiped aside: With the mediator, the child discovers again the rectangular parallelopiped. . . we recall that the base is NOT A SQUARE, but a rectangle.

4. We can compose the formula from this equivalent figure.
   a) $A_{Rp} = A_b \times h$ . . .but because we have a rhombic parallelopiped, we must
   b) Identify that (b) in the first term, using the nomenclature of the rhombus: show the diagonals of the rhombus base. They form the sides of the rectangle base of the new figure: $b_R = d \times \frac{D}{2}$ Sensorially, we again build the rhombic parallelopiped to verify the terms.
   c) So we compose the formula:

$$V_{Rp} = (d \times \frac{D}{2}) \times h \text{ OR } A_b \times h$$

Presentation #3: The Volume of the Regular Right Hexagonal Prism

1. Introduce the whole solid (from Box #1) and IDENTIFY the BASES: regular hexagons. Show the largest circle from the plane figures and the regular hexagon plane figure in relationship to the base of the solid:



. . .the hexagon base is circumscribed about this circle of 10 cm. diameter.

. . .the hexagon (with a diagonal through the center of 10 cm.) is inscribed in the hexagon base.

NOTE: A Sensorial Preparation here for the identification of the apothem.

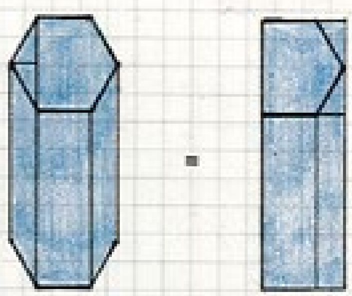
2. Give the nomenclature:

2. This is a regular (the base being a regular polygon). . .right (the lateral surfaces are perpendicular to the plane). . . . . hexagonal (base) prism.

3. Introduce the mediator and describe its division, SHOWING CONGRUENCE WITH THE FIRST.

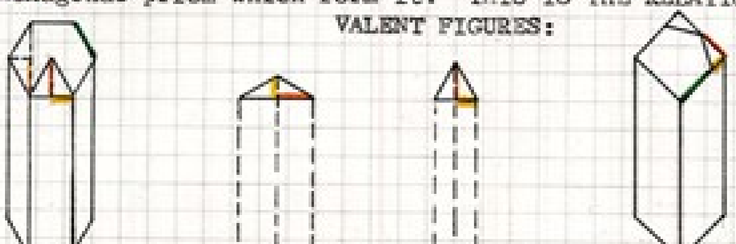
3. To calculate the volume of this figure, we must introduce a mediator. This congruent solid, another regular right hexagonal prism, has been divided first by a line joining non-consecutive vertices. (That is, skipping ONE vertex) The resulting isosceles obtuse-angled triangle gives us a right triangular prism. Then that prism has been cut along its only interior altitude. . .and we obtain two smaller right triangular prisms.

4. Put the whole hexagonal prism aside and the child works with only the mediator. He must transform the prism into one with a rectangular base (that is, a base with perpendicular vertices) so that he can achieve that solid which is divisible into cubes. ALLOW THE CHILD TO EXPLORE THE POSSIBILITIES AND FIND THE SOLUTION. . . which is, finally, the square-based rectangular solid.



. . .we have transformed the regular right hexagonal prism into an equivalent rectangular prism.

5. Using the two small triangular prisms, show the equilateral triangle of which the hexagon base is built. Identify the side of the hexagon as the base of this triangle, that is, each of the small triangular prisms has a side which corresponds to 1/2 the side of the hexagon. . .AND identify the altitude of this triangle as the APOTHEM; that is, the apothem is equal to 1/2 the diagonal traced between non-consecutive vertices. Then identify the lines of the rectangular prism in terms of the lines of the hexagonal prism which form it: THIS IS THE RELATIONSHIP OF LINES IN EQUIVALENT FIGURES:



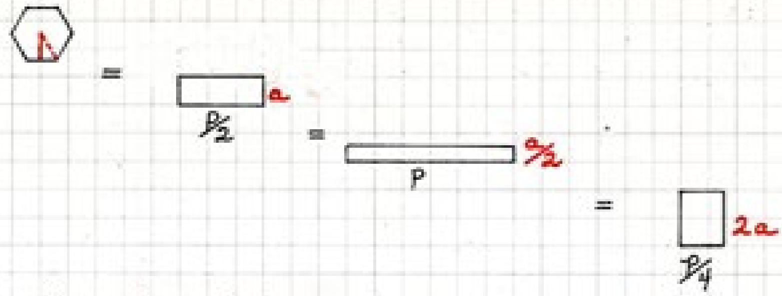
$b_{Rp} = 2a_{Hp}$

$h_{Rp} = \frac{1}{6} p_H + \frac{1}{12} p_H = \frac{3}{12} p_H = \frac{1}{4} p_H$

6. STATEMENT OF EQUIVALENCE: The hexagon which forms the base of this prism is equivalent to that rectangle which has a base equal to two times the apothem of that hexagon and a height equal to 1/4 the perimeter of that hexagon.

NOTE: We have now developed corollary of that original formula which we developed for the area of the regular polygon. . .AND we have, at the same time, found a new rectangle which represents this corollary:

$$A_{RP} = \frac{p \cdot a}{2}$$
$$= \frac{p}{2} \cdot a$$
$$= p \cdot \frac{a}{2}$$
$$= \frac{p}{4} \cdot 2a$$



7. Formulate the formula for the volume of the regular right hexagonal prism:

$V_{Hp} = (p/4 \text{ X } 2a) \text{ X } h$ OR $V_{Hp} = A_b \text{ X } h$

PYRAMIDS: The Second Family

Materials

1. From Box #1, the rectangular parallelopiped and the triangular prism.
2. The solids from Box #2.
3. The "hollow" solids of Box #3. (rectangular parallelopiped, second short rectangular parallelopiped, h = 1/3, and square-based pyramid.)
4. The cube divided into three equal pyramids.
5. Blue cardboard.
6. Sand

Presentation #1: The Volume of the Right Square Pyramid

1. Showing the new figure, the right square pyramid, give the etymology and the nomenclature. Point out the characteristics of the figure: bases and lateral faces.

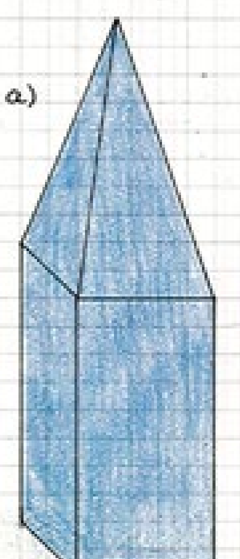
1. This is a pyramid. . .its origin takes us back to Greece and then back to Egypt. From "pyre" meaning "fire." This figure has the shape of a bunch of sticks prepared to be burned.

What kind of a pyramid is this? a right. . .because if we had a plumb line passing through the vertex to the base, the line would coincide with the center of the base. . . . . .that is, the height of the pyramid. . . square. . .the figure rests on only one base (the characteristic of the pyramid family; the other end being a poing). . .and that base is a square. . .pyramid.

This is a right square pyramid.

2. First construction of the formula: SENSORIAL by analogy.
   a) Take the rectangular parallelopiped and verify that both figures have the same base and height.
   b) Take the two corresponding hollow figures and verify congruence with the two solids.
   c) Take a bucket full of dry sand, a knife and paper. Fill the hollow pyramid exactly and fill the hollow rectangular parallelopiped: it requires THREE TIMES the amount of sand contained in the pyramid, so the process must be repeated three times with exactness.
   d) Conclusion: We need three times the quantity of sand in the pyramid to fill the prism. . . so the volume of the prism is 3 X that of the pyramid OR ∴ the volume of the pyramid is equal to 1/2 the volume of the prism.

a)



$$V_{Spy} = \frac{V_{Rp}}{3}$$

Volume. . .
Pyramids: Presentation #1: Square Pyramid. . .
Sensorial proof. . .

e) Take the short rectangular parallelopiped and the corresponding hollow figure; verify that they are congruent, with the same base and height.

f) With the SOLID SHORT PRISM, measure its height off of the height of the tall solid prism: $h_{sp} = 1/3\ h_{tp}$ The height of the short prism is equal to one-third the height of the tall prism.

g) Note that the SAME RELATIONSHIP EXISTS BETWEEN THE TWO CORRESPONDING HOLLOW PRISMS.

h) Now fill the short hollow prism with sand and pour it into the pyramid: we discover that the same sand can be contained in both figures.

i) Conclusions: The pyramid is equivalent to that prism having the same base as the pyramid and 1/3 the height of the pyramid. That is, the right square pyramid is equivalent to the short prism.

j) Thus, starting with the tall prism as our reference, we can construct this formula for the volume of the pyramid:

$$V_{Spy} = A_b \times h/3$$

3. Second proof: With the special prism, the trisected cube.

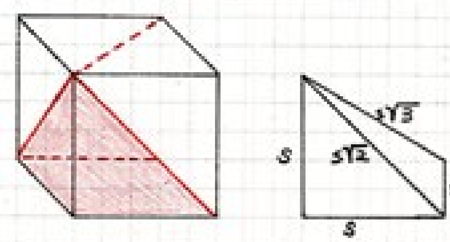
a) Introduce the largest cube of the pink tower as a special prism, a special rectangular parallelopiped. The faces are all squares! (each square a 10 cm^2)

b) Introduce the trisected cube, first verifying congruence with the first: This is our mediator.

c) Dissemble the cube, noting that THE THREE PYRAMIDS WHICH COMPOSE IT ARE EQUAL AMONG THEMSELVES.

d) Identify the base and height of the pyramids as equal to those of the cube.

e) Therefore, we can say that the volume of the pyramid is equal to one-third the volume of the cube having the same base and height. Because this cube is formed of three equal pyramids, each with a base and height equal to that of the cube or prism.



f) Formula: $V_{py} = \frac{V_p}{3}$

OR

$$V_{py} = \frac{A_b \times h}{3}$$

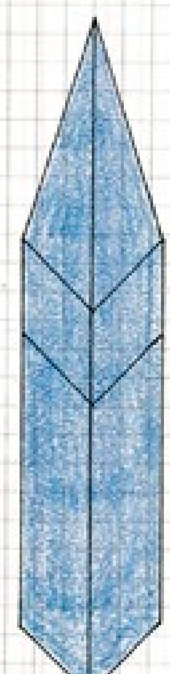
g) Identify the lines: first of the cube (cube edges are the sides of the squares that form it.) Then, the special right-angled isosceles and scalene triangles that form the lateral surfaces of the pyramid.

Presentation #2: The Volume of the Regular right Triangular Pyramid

1. Introduce the triangular pyramid and give the nomenclature: this is a regular (note equilateral base) right triangular pyramid.

2. Compare the tall and short prisms (triangular). Note that the height of the short prism is 1/3 the height of the tall, measuring off as before.

3. Build a steeple with all three figures (see figure), noting that all three have equal bases.



4. Now the child will easily deduce the relationship of 1:3 between the tall prism and the short prism and between the tall prism and the pyramid. . .

5. concluding: The regular right triangular prism is equivalent to that triangular prism having a base equal to the base of the pyramid and a height equal to one-third the height of the pyramid.

then: $V_{Tpy} = A_b \times \frac{h}{3}$

Volume. . .

SOLIDS OF ROTATION: The Third Family

Presentation #1: The Volume of the Cylinder

Material
1. The regular right hexagonal prism, box #1.
2. The right circular cylinder, box #2.

1. Give the nomenclature and etymology of the cylinder. Show the new solid simultaneously with the regular right hexagonal prism. ROLL first the hexagon (a poor roller) and then ROLL THE CYLINDER.

1. This is a cylinder. The word cylinder means "envelops." . ."surrounds all." It also means "it rolls." What is the shape of the base? So this is a circular cylinder. And because the line traced from the center of one base to the center of the lower base forms a perpendicular with the plane, it is a right circular cylinder.

2. Compare the lateral faces of the two prisms: the concept of the cylinder as the limit of the regular prisms. ROLL THE TWO AGAIN TO EMPHASIZE THE CONCEPT.

2. How many faces does the regular right hexagonal prism have? 6 Imagine a prism of 100 faces, then 200, to an infinite number of faces. . .until we come to the cylinder, the limit of the solid prisms. The cylinder rolls very fast.

3. How will we calculate the volume of the cylinder? The formula.

3. I know how to calculate the volume of this prism: $V_{Cp} = b \times h$. But here the base is a circle. . .

So. . .$V_{Cp} = (\pi r^2) \times h$ OR $V_{CP} = A_b \times h$

Presentation #2: The Volume of the Cone

1. Nomenclature and etymology. Identify base, height, point of the cone.

1. This is a right circular cone. The base is circular. . .and that line dropped from the vertex would pass thru the center of the base and form a perpendicular with the plane. "Cone" comes from two words: one meaning "pine cone (coniferous)" and the other from "hone" which is a stone with this shape once used to sharpen knives.

2. Introduce the SHORT CYLINDER: identify the figure. Identify the base as equal to the base of the cone. Show also the TALL CYLINDER. Build a steeple with the three, identifying equal bases. Then measure the height of the short cylinder with that of the tall cylinder: 1/3 h.

3. Show the two cylinders, reidentifying them as the limit of the prisms. Then show the THREE PYRAMIDS, the triangular, the square and finally the cone, and identify the cone as the limit of the pyramids. (It is a right pyramid, the base having an infinite number of sides.)

4. Conclusions: Knowing the volume of the cylinder, I can calculate the volume of the cone because the cone has the same relationship to the cylinder that the pyramid has to the prism. So I can organize the formula in the same way: the volume of the cone is equal to the volume of that cylinder having an equal base and height to the cone DIVIDED BY THREE.

$$V_{cone} = \frac{A_b \times h}{3}$$

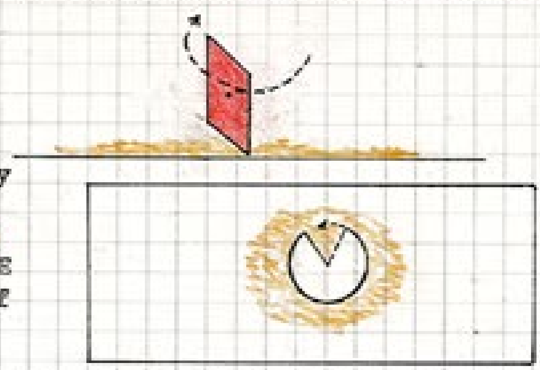
And. . .we must remember that the base of the figure is the circle, so again the calculation for the area of the base must be πr^2.

5. Demonstrate the equivalence of the right circular cone and the short cylinder.
NOTE: If these pieces are constructed of the same wood, they will have equal weight, an interesting and important part of the demonstration.

Volume. . .

Presentation #3: Formation of the Solids

Materials
1. Paper. Heavy blue cardboard from which we cut the figures of the ellipse and oval.
2. Sand.
3. The plane insets: from the metal fractional insets, the rectangle formed by the median in the square divided into halves. The half from the circles. The half of the equilateral triangle.



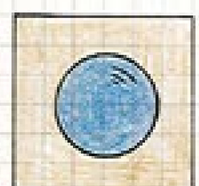
1. Pour some sand on the paper.

2. Take first the rectangle and and hold it vertically in the midst of the sand: identify the axis and the generatrix, that edge which generates the surface of the solid: here the cylinder. Rotate the piece in the sand. NOTE the RESULTING PATTERN IN THE SAND, the base of the cylinder.

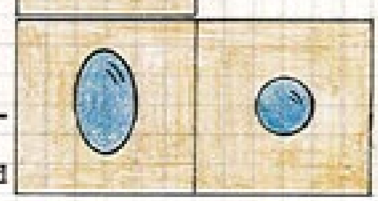
3. Repeat the experience with the right-angled scalene triangle from the metal insets ($\frac{1}{2}$ the equilateral triangle.) The generatrix generates the cone; here the generatrix is the hypotenuse of the figure. The long leg is the axis of rotation. The short leg gives the base pattern in the sand. The point of the triangle becomes the vertex of the cone.

4. INTRODUCE THE THREE CIRCULAR SOLIDS: the sphere, the ellisoid and the ovoid.
   a) Using the half of the circle from the metal insets, rotate the figure in the sand, the diameter now being the axis of rotation, the semi-circumference the generatrix. NOMENCLATURE: "Sphere" means "ball." We note that the rotation of the half circle generates the solid of the sphere.
   b) Using the frame of the ellipse, cut the ellipse from stiff cardboard, then fold the cut-out figure on the major axis and cut in half. Repeat the rotation exercise in the sand with the half-ellipse, using the major axis as the axis of rotation.. NOMENCLATURE: The ellipsoid is called "prolate spheroid," meaning that it is a relative of the sphere, but something happened to that sphere---it was stretched.
   c) Trace the oval on paper in the same way, cutting on the major axis and using one half to show the generation of the ovoid in space.

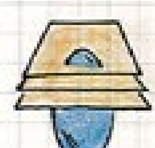
5. Display the three circular solids, their frames and the inset figures from the cabinet. Demonstrate the diameter(s) by showing the solids in the plane inset frames:



   a) The circle of this frame represents the maximum circle of this sphere. Casting a light on this solid would give us this circle shadow.



   b) One picture of the ellipsoid would be that one we see when we show the ellipsoid in the ellipse frame. But we can also get a second picture: showing the ellipse (by the minor axis position) in the frame of the 6 cm. diameter circle frame. The picture from this end will be a circle of 6 cm. diameter.

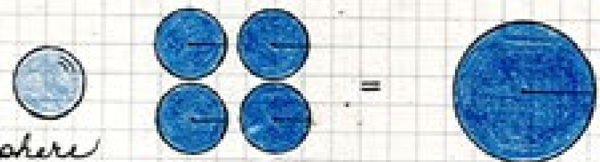


   c) By showing a series of frames (using the frames of the circles from the circle drawer) on the ovoid, we see that we could have many pictures of this solid. There are circles of an infinite number of different diameters represented in this solid. . .SO we cannot have a fixed rule for each oval or ovid when calculating the area and the volume.

Volume. . .
Solids of Rotation. . .

Presentation #4: The Volume of the Sphere

1. Introducing the problem: To find the volume of the sphere, we must know the area (A) of the spherical surface. . .because we can observe that there is no differentiation between the faces and the bases. Archimedes discovered for the first time the volume of the sphere. . .and the area of the spherical surface. This was confirmed later by Galileo, and then by his disciple Bonaventura Cavalieri during the first half of the 17th century. Cavalieri was a professor at Bologna. And it is from his work that the theory of the area of the spherical surface takes its name.
These mathematicians discovered that the area of the spherical surface is equal to the area of the maximum circle of the sphere times 4. . .OR. .$A_{ss} = 4 \quad r^2$.
This maximum circle times 4 gives gives us the area of the spherical surface of that sphere having the same diameter as each circle. AND. . .those four circles give a new circle which is four times the area of one of them.
INTRODUCE CARDBOARD CIRCLES AS SHOWN:



sphere $r = 5$ — spherical surface $= 4\pi(5)^2$ = $A_o = \pi r^2$, $r = 10$

$A_{largest\,o} \times 4$ = spherical surface
or... $\pi r^2 \times 4 = 4\pi(5)^2$
then 314 = spherical surface
And to determine (r) of the new circle:
$314 = \pi r^2$
$100 = r^2$
$10 = r$

The radius of the new circle is 2 × the radius of the small.
That is, the ratio of linear measure is 1:2.

NOTE: We can prove this equivalence as shown above (four circles equivalent to the larger one) or we can prove the equivalence by weight. Then measure both radii.

2. INTRODUCE THE SHORT CONE, that cone with a base equal to the spherical surface of the sphere and a height equal to the radius of the sphere.

$h = r$
b = spherical surface

Note that the new cone (wooden neutral) has a base that is four times the area of the 10 cm. diameter circle, the largest circle of the sphere.

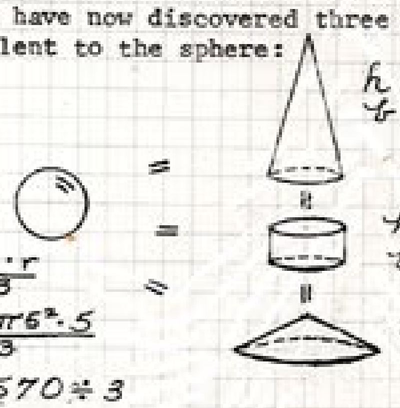
2. We calculate the volume of the sphere with the formula we have developed for the pyramids and the cone:

$$V_S = \frac{\text{spherical surface} \times r}{3}$$

BECAUSE the sphere is equivalent to this cone.
Why? The new cone has a base equivalent to the spherical surface of the sphere (b) and a height (h) equal to the radius of the sphere.

3. Concluding the formula: $V_S = \dfrac{4 \quad r^2 \times r}{3}$ OR $V_S = \dfrac{4}{3} \quad r^3$

4. We have now discovered three solids equivalent to the sphere:



$r = 5$

$V_{sphere} \cong \dfrac{S \cdot r}{3}$
$= \dfrac{4\pi 5^2 \cdot 5}{3}$
$= 1570 \div 3$
$= 523.\bar{3}$

$h = 20 \; (4 \times r)$
$b = \pi r^2 \; (r = 5)$

$h = \frac{20}{3} = 6.67$
$b = \pi r^2 \; (r = 5)$

$h = r \; (5)$
$b = 4\pi r^2$

$V_{cone} = \dfrac{\pi r^2 \cdot h}{3}$
$= \dfrac{3.14(5^2) \cdot 20}{3} = \dfrac{1570}{3} = 523.\bar{3}$

$V_{cyl} = \pi r^2 \cdot h = \pi \cdot (5^2) \cdot 6.67$
$= 3.14 \times 25 \times 6.67 = 523.5\ldots$

$V_{cone} = \dfrac{4\pi r^2 \cdot r}{3} = \dfrac{4\pi(5^2) \cdot 5}{3}$
$= 523.\bar{3}$

NOTE: The two cones are equivalent because they have the same proportions, but those dimensions are inverted. We have already proven the equivalence between the tall cone and the short cylinder.

Volume. . .
The Volume of the Sphere. . .

The secret of the equivalence transformation from the sphere to the short cone is in the regular polyhedrons. We have five regular polyhedrons: tetrahedron (four equilateral triangles), cube (hexahedron) (six squares), octahedron (eight equilateral triangles), dodecahedron (twelve pentagons), and the icosahedron (twenty equilateral triangles.)

The icosahedron is composed then of 20 faces (twenty equilateral triangles). . .OR. . . twenty regular right triangular pyramids. The height of each pyramid will be the distance from the mid-point of the side to the center (the altitude of the pyramid.) Then, if we opened that icosahedron so that each of the faces became a base on the same plane, we would have a pattern of pyramid vertices with which we could show a proof similar to the proof of the regular polygon triangle conversion. That is, connecting one vertex to the base of each pyramid, we could transform that volume into the volume of a single pyramid.

Then, if we have an icosahedron of 100, 200, 1,000,000 pyramids, we begin to approach the limit of such a proof. Until we have the sphere which is the limit of those polyhedrons with an infinite number of faces. Each point on that spherical surface is then the base of a hypothetical pyramid or a cone. And we can transform this regular hedron in the same way, with the resulting short cone of our proof. The cone will have a base equal to the spherical surface and a height equal to the radius of the sphere.

Going on from here. . .to the Volume of the Ellipsoid and the Ovoid. . .

to Total and Lateral Surfaces of the Solids.

The Study of the Five Regular Polyhedrons

Analysis of the Regular Polyhedrons and the Calculation of their Volume

and Chapter VI. Relationships